Exercise 4.14

Prove that

$$\lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} f''(W(t_j)) \left[W(t_{j+1}) - W(t_j)\right]^2 = \int_0^T f''(W(t)) \mathrm{d}t. \tag{1}$$

The proof is in three steps:

1. Define $Z_j = f''(W(t_j)) \Big[(W(t_{j+1}) - W(t_j))^2 - (t_{j+1} - t_j)\Big]$. Show Z_j is $\mathcal{F}(t_{j+1})$-measurable and

$$\mathbb{E}[Z_j|\mathcal{F}(t_j)] = 0 \quad \text{and} \quad \mathbb{E}[Z_j^2|\mathcal{F}(t_j)] = 2\left[f''(W(t_j))\right]^2 (t_{j+1} - t_j)^2$$

2. Under the assumption that $\mathbb{E}\int_0^T \left[f''(W(t))\right]^2 \mathrm{d}t < +\infty$, show that $\lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} Z_j = 0$.
3. Conclude Eq (1).

Proof

1. We have that

$$\begin{aligned} f''(W(t_j))\mathbb{E}[Z_j|\mathcal{F}(t_j)] &= f''(W(t_j))\mathbb{E}[(W(t_{j+1}) - W(t_j))^2 - (t_{j+1} - t_j)|\mathcal{F}(t_j)] \\ &= f''(W(t_j)) \cdot (\mathrm{Var}(W(t_{j+1}) - W(t_j)) - (t_{j+1} - t_j)) = 0. \end{aligned}$$

To show the second inequality, recall that the kurtosis of normal random variables is 3. Hence, since $W(t_{j+1}) - W(t_j)$ is normal

$$\frac{\mathbb{E}\left[(W(t_{j+1}) - W(t_j))^4\right]}{\mathrm{Var}\left(W(t_{j+1}) - W(t_j)\right)^2} = 3$$

Therefore, $\mathbb{E}\left[(W(t_{j+1}) - W(t_j))^4\right] = 3\left(t_{j+1} - t_j\right)^2$. Thus,

$$\begin{aligned} \mathbb{E}[Z_j^2|\mathcal{F}(t_j)] &= \left[f''(W(t_j))\right]^2 \cdot \Big(3\left(t_{j+1} - t_j\right)^2 - 2\left(t_{j+1} - t_j\right)^2 + \left(t_{j+1} - t_j\right)^2\Big) \\ &= 2\left[f''(W(t_j))\right]^2 \left(t_{j+1} - t_j\right)^2. \end{aligned}$$

2. We begin by noting that $\mathbb{E}Z_j = \mathbb{E}\left[\mathbb{E}[Z_j|\mathcal{F}(t_j)]\right] = 0$. Thus $\mathbb{E}\sum_{j=0}^{n-1} Z_j = 0$. Note that

$$\mathcal{F}(t_0) \subseteq \mathcal{F}(t_1) \subseteq \cdots \subseteq \mathcal{F}(t_{n-1}) \subseteq \mathcal{F}(t_n) = \mathcal{F}(T).$$

Denote by $S_k := \sum_{j=0}^k Z_j$. Notice that S_{n-2} is $\mathcal{F}(t_{n-1})$-measurable. We have that

$$\begin{aligned} \mathrm{Var}\, S_{n-1} &= \mathbb{E}[S_{n-2} + Z_{n-1}]^2 \\ &= \mathbb{E}\left[\mathbb{E}\left[[S_{n-2} + Z_{n-1}]^2|\mathcal{F}(t_{n-1})\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[S_{n-2}^2 + 2S_{n-2}Z_{n-1} + Z_{n-1}^2|\mathcal{F}(t_{n-1})\right]\right] \\ &= \mathbb{E}\left[S_{n-2}^2 + 2S_{n-2}\underbrace{\mathbb{E}\left[Z_{n-1}|\mathcal{F}(t_{n-1})\right]}_{=0} + \mathbb{E}\left[Z_{n-1}^2|\mathcal{F}(t_{n-1})\right]\right] \\ &= \mathbb{E}\left[S_{n-2}^2\right] + \mathbb{E}\left[\mathbb{E}\left[Z_{n-1}^2|\mathcal{F}(t_{n-1})\right]\right]. \end{aligned}$$

Inductively, we obtain that

$$\operatorname{Var} S_{n-1} = \sum_{j=0}^{n-1} \mathbb{E}\left[Z_j^2\right].$$

Therefore,

$$\operatorname{Var} S_{n-1} = 2\mathbb{E} \sum_{j=0}^{n-1} \left[f''(W(t_j))\right]^2 (t_{j+1} - t_j)^2 \le \|\Pi\| \cdot \mathbb{E} \int_0^T \left[f''(W(t))\right]^2 \mathrm{d}t.$$

By assumption, $\mathbb{E} \int_0^T \left[f''(W(t))\right]^2 \mathrm{d}t < +\infty$. Thus, $\operatorname{Var} S_{n-1} \to 0$ as $\|\Pi\| \to 0$. Putting pieces together, we obtain that

$$\lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} Z_j = 0$$