Exercise 4.15 (Creating correlated Brownian motions from independent ones)

Consider a d-dimensional Brownian motion $\mathcal{W}_d(t) = (W_1(t), \cdots, W_d(t))$ and let $\Sigma := (\sigma_{ij}(t))_{1\le i\le m, 1\le j\le d}$ be adapted processes w.r.t the filtration generated by these Brownian motions. Define

$$\sigma_i(t) = \left[\sum_{j=1}^{d} \sigma_{ij}^2(t)\right]^{\frac{1}{2}}$$

By assumption $\sigma_i(t) > 0$. Define

$$\tilde{\Sigma}_{i,j}(t) = \frac{\Sigma_{i,j}(t)}{\sigma_i(t)}$$

Note that each row of $\tilde{\Sigma}(t)$ has 2-norm equal to 1. Define

$$B_i(t) = \int_0^t \tilde{\Sigma}_{i,:}(s)^T \mathrm{d}\mathcal{W}_d(s)$$

Here $A_{i,:}$ denotes the i-th row of matrix A.

1. Show that $B_i(t)$ is a Brownian motion.
2. $\mathrm{d}B_i(t)\mathrm{d}B_k(t) = \rho_{ik}(t)\mathrm{d}t$ where
$$\rho_{ik}(t) = \tilde{\Sigma}_{i,:}(t)^T \tilde{\Sigma}_{k,:}(t)$$

Proof

1. $B_i(t)$ is a martingale since it is the sum of Itô integrals which are martingales. Notice that
$$\mathrm{d}B_i(t) = \tilde{\Sigma}_{i,:}(t)^T \mathrm{d}\mathcal{W}_d(t)$$
Denoting by I_d, the $d \times d$ identity matrix, the result follows using Levy's theorem as
$$\mathrm{d}B_i(t)\mathrm{d}B_i(t) = \tilde{\Sigma}_{i,:}(t)^T \underbrace{\mathrm{d}\mathcal{W}_d(t)\mathrm{d}\mathcal{W}_d(t)^T}_{=\mathrm{d}t\cdot I_d} \tilde{\Sigma}_{i,:}(t) = \mathrm{d}t \cdot \tilde{\Sigma}_{i,:}(t)^T \tilde{\Sigma}_{i,:}(t) = \mathrm{d}t$$
2. The result immediately follows since
$$\mathrm{d}B_i(t)\mathrm{d}B_i(t) = \tilde{\Sigma}_{i,:}(t)^T \underbrace{\mathrm{d}\mathcal{W}_d(t)\mathrm{d}\mathcal{W}_d(t)^T}_{=\mathrm{d}t\cdot I_d} \tilde{\Sigma}_{k,:}(t)$$