Exercise 4.16 (Creating independent Brownian motions from correlated ones)

Consider d Brownian motions satisfying

$$\mathrm{d}B_i(t)\mathrm{d}B_k(t) = \rho_{ik}(t)\mathrm{d}t$$

Here $\rho_{ik}(t)$ are adapted processes satisfying $-1 < \rho_{ik}(t) < 1$ when $i \neq k$. Also, $\rho_{ik}(t) = 1$.

Assumption: The following symmetric matrix is positive definite:

$$C(t)_{i,j} := \rho_{ij}(t)$$

By definition, $\rho_{ij}(t) = \rho_{ji}(t)$. Find matrix $A(t)$ and d-dimensional Brownian motion $\mathcal{W}_d(t) = (W_1(t), \cdots, W_d(t))$ with independent entries such that

$$B_i(t) = \int_0^t A_{i,:}(t)^T \mathrm{d}\mathcal{W}_d(t)$$

Proof

Denote by $\mathcal{B}_i(t) := B_i(t)$. $\mathcal{B}$ is a d dimensional random vector which satisfies

$$\mathrm{d}\mathcal{B}_i(t)\mathrm{d}\mathcal{B}_i(t)^T = C(t)\mathrm{d}t.$$

Since $C(t) \succ 0$, there exists non-singular $d \times d$ matrix $C_0(t)$ such that $C(t) = A(t)A(t)^T$.[1] Therefore, letting $C_0(t) := A(t)^{-1}$, it follows that

$$\mathrm{d}\mathcal{B}_i(t)\mathrm{d}\mathcal{B}_i(t)^T = A(t)A(t)^T\mathrm{d}t \Rightarrow C_0(t)\mathrm{d}\mathcal{B}_i(t)\mathrm{d}\mathcal{B}_i(t)^T C_0(t)^T = I_d \cdot \mathrm{d}t.$$

Denote by

$$W_i(t) := \int_0^t {C_0}_{i,:}^T(s)\mathrm{d}\mathcal{B}(s)$$

Since $W_i(t)$ is sum of Itô integrals, they are martingale. On the other hands,

$$\mathrm{d}\mathcal{W}_d(t) = C_0(t)^T\mathrm{d}\mathcal{B}(t)$$

Therefore,

$$\begin{aligned}\mathrm{d}\mathcal{W}_d(t)\mathrm{d}\mathcal{W}_d(t)^T &= C_0(t)^T\mathrm{d}\mathcal{B}(t)\mathrm{d}\mathcal{B}(t)^T C_0(t)^T \\ &= C_0(t)^T A(t)A(t)^T C_0(t)^T\mathrm{d}t \\ &= I_d\mathrm{d}t.\end{aligned}$$

Thus, by Levy theorem W_i for $i = 1, \cdots, d$ are independent Brownian motion. Finally, we have that

$$\mathrm{d}\mathcal{B}(t) = A(t)^T\mathrm{d}\mathcal{W}_d(t).$$

The result immediately follows.

[1] From linear algebra, we know that $A(t)$ could be chosen in such a way that it is also an adapted process!