Exercise 4.17 (Instantaneous correlation)

Consider the following Itô processes

$$X_1(t) = X_1(0) + \int_0^t \Theta_1(u)\mathrm{d}u + \int_0^t \sigma_1(u)\mathrm{d}B_1(u),$$
$$X_2(t) = X_2(0) + \int_0^t \Theta_2(u)\mathrm{d}u + \int_0^t \sigma_2(u)\mathrm{d}B_2(u).$$

Here $B_1(t), B_2(t), t \geq 0$ are Brownian motions satisfying $\mathrm{d}B_1(t)\mathrm{d}B_2(t) = \rho(t)\mathrm{d}t$. We also assume that for some constant M, the following bounds hold almost surely for all $t \geq 0$.

$$|\sigma_1(t)|, |\sigma_2(t)|, |\Theta_1(t)|, |\Theta_2(t)|, |\rho(t)| \leq M.$$

Show that

$$\lim_{t \downarrow t_0} \frac{C(t)}{\sqrt{V_1(t)V_2(t)}} = \rho(t_0). \tag{1}$$

Here $V_i(\epsilon)$ denotes variance of $X_i(t_0+\epsilon) - X_i(t_0)$ conditioned on $\mathcal{F}(t_0)$. Also, $C(\epsilon)$ denotes the covariance between $X_1(t_0+\epsilon) - X_1(t_0)$ and $X_2(t_0+\epsilon) - X_2(t_0)$ conditioned on $\mathcal{F}(t_0)$. In view of Eq (1), $\rho(t)$ is called the instantaneous correlation between $X_1(t)$ and $X_2(t)$.

Proof

We first show Eq (1) holds when $\rho, \Theta_1, \Theta_2, \sigma_1, \sigma_2$ are constant. In this case,

$$X_1(t) = X_1(0) + \Theta_1 t + \sigma_1 B_1(t), \; X_2(t) = X_2(0) + \Theta_2 t + \sigma_2 B_2(t).$$

Note that

$$\mathbb{E}[\underbrace{X_i(t_0+\epsilon) - X_i(t_0)}_{:=U_i(t)} | \mathcal{F}(t_0)] = \Theta_i \epsilon.$$

It is emphasized that $U_i(t)$ is only defined for $t \geq t_0$. Moreover,

$$\begin{aligned}
\mathbb{E}[U_i^2(t)|\mathcal{F}(t_0)] &= \mathbb{E}[(\Theta_i\epsilon + \sigma_i(B_i(t) - B_i(t_0)))^2 |\mathcal{F}(t_0)] \\
&= \Theta_i^2\epsilon^2 + 2\sigma_i\Theta_i\epsilon\mathbb{E}[B_i(t) - B_i(t_0)|\mathcal{F}(t_0)] + \sigma_i^2\mathbb{E}[(B_i(t) - B_i(t_0))^2 |\mathcal{F}(t_0)] \\
&= \Theta_i^2\epsilon^2 + \sigma_i^2\epsilon.
\end{aligned}$$

Here we used the fact that $B_i(s+t_0) - B_i(t_0), s \geq 0$ is a Brownian motion independent of $\mathcal{F}(t_0)$; We will use it again when computing the covariance between $U_1(t)$ and $U_2(t)$. We have that

$$\mathbb{E}[U_1(t)U_2(t)|\mathcal{F}(t_0)] = \Theta_1\Theta_2\epsilon^2 + \sigma_1\sigma_2\mathbb{E}[(B_1(t) - B_1(t_0))(B_2(t) - B_2(t_0))|\mathcal{F}(t_0)]$$

On the other hand,

$$\begin{aligned}
(B_1(t) - B_1(t_0))(B_2(t) - B_2(t_0)) = &\int_0^\epsilon (B_1(t_0+u) - B_1(t_0))\,\mathrm{d}B_2(u) \\
&+ \int_0^\epsilon (B_2(t_0+u) - B_2(t_0))\,\mathrm{d}B_1(u) \\
&+ \int_0^\epsilon \mathrm{d}B_1(u)\mathrm{d}B_2(u).
\end{aligned}$$

Therefore,

$$\mathbb{E}[(B_1(t_0+\epsilon)-B_1(t_0))\,(B_2(t_0+\epsilon)-B_2(t_0))] = 0+0+\rho\epsilon = \rho\epsilon.$$

Note that using the Independence Lemma, we have that

$$\mathbb{E}[(B_1(t)-B_1(t_0))\,(B_2(t)-B_2(t_0))] = \mathbb{E}[(B_1(t)-B_1(t_0))\,(B_2(t)-B_2(t_0))\,|\mathcal{F}(t_0)].$$

Putting pieces together, we conclude that

$$\text{Cov}\,(U_1(t),U_2(t)) = \frac{\rho\sigma_1\sigma_2\epsilon}{\sqrt{\sigma_1^2\epsilon\cdot\sigma_2^2\epsilon}} = \rho.$$

We now consider the general case. Notice that

$$U_i(t) = \int_{t_0}^t \Theta_i(u)\mathrm{d}u + \int_{t_0}^t \sigma_i(u)\mathrm{d}B_i(u).$$

Taking expectations from both sides gives

$$\mathbb{E}\left[U_i(t)|\mathcal{F}(t_0)\right] = \mathbb{E}\left[\int_{t_0}^t \Theta_i(u)\mathrm{d}u|\mathcal{F}(t_0)\right]$$

Dominated convergence theorem for conditional expectations implies that

$$\lim_{t\downarrow t_0}\frac{1}{t-t_0}\mathbb{E}\left[\int_{t_0}^t \Theta_i(u)\mathrm{d}u|\mathcal{F}(t_0)\right] = \mathbb{E}\left[\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\int_{t_0}^{t_0+\epsilon} \Theta_i(u)\mathrm{d}u|\mathcal{F}(t_0)\right] = \Theta_i(t_0).$$

Here we used the assumption that $|\Theta_i(t)|\le M$ almost surely. Indeed,

$$\left|\frac{1}{t-t_0}\int_{t_0}^t \Theta_i(u)\mathrm{d}u\right| \le \frac{1}{t-t_0}\int_{t_0}^t \left|\Theta_i(u)\right|\mathrm{d}u \le M.$$

We therefore have that

$$\lim_{t\downarrow t_0}\frac{1}{t-t_0}\mathbb{E}\left[U_i(t)|\mathcal{F}(t_0)\right] = \Theta_i(t_0).$$

$$U_i(t)U_j(t) = \int_{t_0}^t U_i(u)\mathrm{d}U_j(u) + \int_{t_0}^t U_j(u)\mathrm{d}U_i(u) + \int_{t_0}^t \mathrm{d}U_j(u)\mathrm{d}U_i(u)$$

By definition of integrals w.r.t. Itô processes,

$$\int_{t_0}^t U_i(u)\mathrm{d}U_j(u) = \int_{t_0}^t U_i(u)\Theta_j(u)\mathrm{d}u + \int_{t_0}^t U_i(u)\sigma_j(u)\mathrm{d}B_j(u)$$

Thus,

$$\lim_{t\downarrow t_0}\frac{1}{t-t_0}\mathbb{E}\int_{t_0}^t U_i(u)\mathrm{d}U_j(u) = \lim_{t\downarrow t_0}\frac{1}{t-t_0}\mathbb{E}\int_{t_0}^t U_i(u)\Theta_j(u)\mathrm{d}u = \underbrace{U_i(t_0)}_{=0}\Theta_j(t_0) = 0$$

On the other hand, letting $\rho_{i,j}(u)=\rho(u)$ whenever $i\neq j$ and $\rho_{i,i}(u)=1$.

$$\int_{t_0}^t \mathrm{d}U_j(u)\mathrm{d}U_i(u) = \int_{t_0}^t \sigma_i(u)\sigma_j(u)\rho_{i,j}(u)\mathrm{d}u.$$

Thus,

$$\lim_{t\downarrow t_0}\frac{1}{t-t_0}\mathbb{E}\int_{t_0}^{t}\mathrm{d}U_j(u)\mathrm{d}U_i(u)=\lim_{t\downarrow t_0}\frac{1}{t-t_0}\mathbb{E}\int_{t_0}^{t}\sigma_i(u)\sigma_j(u)\rho_{i,j}(u)\mathrm{d}u=\sigma_i(t_0)\sigma_j(t_0)\rho_{i,j}(t_0).$$

Putting pieces together,

$$\lim_{t\downarrow t_0}\frac{1}{t-t_0}\mathbb{E}U_i(t)U_j(t)=\sigma_i(t_0)\sigma_j(t_0)\rho_{i,j}(t_0)$$

Therefore,

$$\begin{aligned}\lim_{t\downarrow t_0}\frac{1}{t-t_0}C(t)&=\lim_{t\downarrow t_0}\frac{1}{t-t_0}\mathbb{E}U_1(t)U_2(t)-\lim_{t\downarrow t_0}\frac{1}{t-t_0}\mathbb{E}U_1(t)\mathbb{E}U_2(t)\\&=\sigma_1(t_0)\sigma_2(t_0)\rho(t_0)-\Theta_1(t_0)U_2(t_0)\\&=\sigma_1(t_0)\sigma_2(t_0)\rho(t_0).\end{aligned}$$

Moreover,

$$\begin{aligned}\lim_{t\downarrow t_0}\frac{1}{t-t_0}V_i(t)&=\lim_{t\downarrow t_0}\frac{1}{t-t_0}\mathbb{E}U_i^2(t)-\frac{1}{t-t_0}\mathbb{E}U_i(t)\mathbb{E}U_i(t)\\&=\sigma_i(t_0)^2-\Theta_i(t_0)\lim_{t\downarrow t_0}\mathbb{E}\int_{t_0}^{t}\Theta_i(u)\mathrm{d}u\\&=\sigma_i(t_0)^2.\end{aligned}$$

Here $\left|\lim_{t\downarrow t_0}\int_{t_0}^{t}\Theta_i(u)\mathrm{d}u\right|\le\lim_{t\downarrow t_0}\int_{t_0}^{t}|\Theta_i(u)|\,\mathrm{d}u\le\lim_{t\downarrow t_0}M(t-t_0)=0$. Finally, putting pieces together, we conclude that

$$\lim_{t\downarrow t_0}\frac{C(t)}{\sqrt{V_1(t)V_2(t)}}=\lim_{t\downarrow t_0}\frac{\frac{1}{t-t_0}C(t)}{\sqrt{\frac{1}{t-t_0}V_1(t)\cdot\frac{1}{t-t_0}V_2(t)}}=\frac{\sigma_1(t_0)\sigma_2(t_0)\rho(t_0)}{\sigma_1(t_0)\sigma_1(t_0)}=\rho(t_0).$$