Exercise 4.18 (State price density process)

Let the stock price be a geometric Brownian motion *i.e.,* $\mathrm{d}S(t) = \alpha S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t)$. Assume r is the interest rate and define the *market price of risk* to be

$$\theta := \frac{\alpha - r}{\sigma}$$

Moreover, define *state price density* to be

$$\zeta(t) = \exp\left(-\theta W(t) - \left(r + \tfrac{1}{2}\theta^2\right) t\right)$$

1. Prove that
$$\mathrm{d}\zeta(t) = -\theta\zeta(t)\mathrm{d}W(t) - r\zeta(t)\mathrm{d}t$$

2. Consider an investor's portfolio process $\Delta(t)$ and let X denotes the porfolio itself. Recall that
$$\mathrm{d}X(t) = \underbrace{rX(t)\mathrm{d}t}_{\text{average rate of return}} + \underbrace{\Delta(t)(\alpha - r)S(t)\mathrm{d}t}_{\text{risk premium}} + \underbrace{\Delta(t)\sigma S(t)\mathrm{d}W(t)}_{\text{volatility term}}.$$
Show that $\zeta(t)X(t)$ is a martingale.

3. Show that the present value of random payment $V(T)$ at time T equals to $\mathbb{E}[\zeta(T)V(T)]$.

Proof

1. Let $Z(t) = -\theta W(t) - \left(r + \frac{1}{2}\theta^2\right) t$ and $f(z) = e^z$. Itô 's lemma gives us
$$\mathrm{d}f(Z(t)) = f'(Z(t))\mathrm{d}Z(t) + \tfrac{1}{2}f''(Z(t))\mathrm{d}Z(t)\mathrm{d}Z(t).$$
Therefore,
$$\begin{aligned}\mathrm{d}\zeta(t) &= \zeta(t)\left(-\theta\mathrm{d}W(t) - \left(r + \tfrac{1}{2}\theta^2\right)\mathrm{d}t\right) + \tfrac{1}{2}\theta^2\zeta(t)\mathrm{d}t \\ &= -\theta\zeta(t)\mathrm{d}W(t) - r\zeta(t)\mathrm{d}t.\end{aligned}$$

2. Notice that
$$\mathrm{d}\zeta(t)X(t) = X(t)\mathrm{d}\zeta(t) + \zeta(t)\mathrm{d}X(t) + \mathrm{d}X(t)\mathrm{d}\zeta(t)$$
We have that
$$\begin{aligned}X(t)\mathrm{d}\zeta(t) &= \zeta(t)\cdot X(t)\cdot[-\theta\mathrm{d}W(t) - r\mathrm{d}t] \\ \zeta(t)\mathrm{d}X(t) &= \zeta(t)\cdot[rX(t)\mathrm{d}t + \Delta(t)(\alpha - r)S(t)\mathrm{d}t + \Delta(t)\sigma S(t)\mathrm{d}W(t)] \\ \mathrm{d}X(t)\mathrm{d}\zeta(t) &= -\theta\zeta(t)\cdot\Delta(t)\sigma S(t)\mathrm{d}t.\end{aligned}$$
Therefore, $\mathrm{d}t$ term in $\mathrm{d}\zeta(t)X(t)$ divided by $\zeta(t)$ equals to
$$-rX(t) + rX(t) + \Delta(t)(\alpha - r)S(t) - \theta\Delta(t)\sigma S(t) = 0.$$
Here we used $\theta\sigma = \alpha - r$.

3. Letting $X(T) = V(T)$, we obtain that
$$X(0) = \zeta(0)X(0) = \mathbb{E}[\zeta(T)X(T)] = \mathbb{E}[\zeta(T)V(T)].$$

Remark In view of risk-neutral pricing (discussed in next chapter), the following is true

$$\mathbb{E}[\zeta(T)X(T)] = \tilde{\mathbb{E}}[e^{-rT}V(T)]$$

$\zeta(t) = e^{-rt}Z(t)$ where $Z(t)$ is the Radon-Nikodym derivative process. See 5.2.11.