Exercise 4.19

1. Let $W(t)$ be a Brownian motion and define

$$B(t) = \int_0^t \operatorname{sgn}(W(s))dW(s)$$

Show that $B(t)$ is a Brownian motion.

2. Compute $\mathrm{d}B(t)W(t)$. Show that $\mathbb{E}[B(t)W(t)] = 0$.

3. Compute $\mathrm{d}B(t)W^2(t)$. Show that $\mathbb{E}[B(t)W^2(t)] \neq \mathbb{E}B(t) \cdot \mathbb{E}W^2(t)$.

Proof

1. $B(t)$ is continuous since the following bound holds:

$$\mathbb{E}\int_0^t \operatorname{sgn}(W(s))^2 ds \leq s < +\infty.$$

Hence Theorem 4.3.1 shows that $B(t)$ is continuous. Indeed, we could show the above integral is exactly equal to s. Observe that

$$A := \{s \in [0,t] : W(s) = 0\} \text{ has Lebesgue measure zero.} \tag{1}$$

To see this, note

$$\begin{aligned} \mu(A) = \int_0^t 1_A(s)\mathrm{d}s \Rightarrow \mathbb{E}(\mu(A)) &= \int_0^t \mathbb{E}(1_A(s))\mathrm{d}s \\ &= \int_0^t \mathbb{P}(W(s) = 0)\mathrm{d}s \\ &= 0. \end{aligned}$$

Here we used Fubini theorem and the fact that indicator is non-negative. Next, we use Levy's Theorem to show that $B(t)$ is a Brownian motion. $B(0) = 0$ and $B(t)$ is a martingale relative to filtration generated by $W(s) : s \leq t$ based on Theorem 4.3.1. Theorem 4.3.1 also gives

$$[B,B](t) = \mathbb{E}\int_0^t \operatorname{sgn}(W(s))^2 \mathrm{d}s = t$$

Here we used (1). By Levy Theorem, $B(t)$ is Brownian motion.

2. We have that

$$\mathrm{d}B(t)W(t) = B(t)\mathrm{d}W(t) + W(t)\mathrm{d}B(t) + \mathrm{d}W(t)\mathrm{d}B(t)$$

Continuing,

$$\mathrm{d}W(t)\mathrm{d}B(t) = \operatorname{sgn}(W(t))\mathrm{d}W(t)\mathrm{d}W(t) = \operatorname{sgn}(W(t))\mathrm{d}t.$$

Moreover,

$$\mathbb{E}\int_0^t B(s)^2 \mathrm{d}s = \int_0^t \mathbb{E}B(s)^2 \mathrm{d}s = \int_0^t \int_0^s \operatorname{sgn}(W(u))^2 \mathrm{d}u\mathrm{d}s = \frac{t^2}{2} < +\infty.$$

And

$$\mathbb{E}\int_0^t W(s)^2 ds = \int_0^t \mathbb{E}W(s)^2 ds = \int_0^t s ds = \frac{t^2}{2} < +\infty$$

Rewriting the Itô product rule above, we have

$$d[B(t)W(t)] = B(t)dW(t) + |W(t)|dW(t) + \operatorname{sgn}(W(t))dt.$$

Taking integrals and then expectations, we have that

$$\begin{aligned}\mathbb{E}B(t)W(t) &= \underbrace{\mathbb{E}\int_0^t B(s)dW(s)}_{:=I_1(t)} + \underbrace{\mathbb{E}\int_0^t |W(s)|dW(s)}_{:=I_2(t)} + \mathbb{E}\int_0^t \operatorname{sgn}(W(s))ds \\ &= 0 + 0 + \int_0^t \underbrace{\mathbb{E}\operatorname{sgn}(W(s))}_{=0} ds = 0.\end{aligned}$$

We used the regularity conditions above to be able to use Theorem 4.3.1 *i.e.,* Itô integrals are martingales and thus $\mathbb{E}I_1(t) = \mathbb{E}I_2(t) = 0$.

3. We begin by noting that

$$\mathrm{d}W^2(t) = W(t)\mathrm{d}W(t) + W(t)\mathrm{d}W(t) + \mathrm{d}W(t)\mathrm{d}W(t) = 2W(t)\mathrm{d}W(t) + \mathrm{d}t.$$

Next,

$$\mathrm{d}B(t)W^2(t) = \underbrace{W^2(t)\mathrm{d}B(t)}_{W^2(t)\operatorname{sgn}(W(t))\mathrm{d}W(t)} + \underbrace{B(t)\mathrm{d}W^2(t)}_{2B(t)W(t)\mathrm{d}W(t)} + \underbrace{\mathrm{d}B(t)\mathrm{d}W^2(t)}_{2W(t)\operatorname{sgn}(W(t))\mathrm{d}t}$$

Consequently,

$$\mathbb{E}B(t)W^2(t) = 2\mathbb{E}\int_0^t |W(s)|\,\mathrm{d}s = 2\int_0^t \mathbb{E}\,|W(s)|\,\mathrm{d}s = 2\sqrt{\tfrac{2}{\pi}}\int_0^t \sqrt{s}\mathrm{d}s > 0.$$

Therefore,

$$\underbrace{\mathbb{E}B(t)W^2(t)}_{>0} \neq \underbrace{\mathbb{E}B(t)}_{=0} \cdot \underbrace{\mathbb{E}W^2(t)}_{=t}$$