Exercise 4.2

Let $W(t), t \geq 0$ be a Brownian motion with the associated filtration denoted by $\mathcal{F}(t)$. Consider a partition of $[0, T]$ as $t_0 < \cdots < t_n$ and assume that $\Delta(t)$ is constant on $[t_j, t_{j+1})$. Δ is assumed to be deterministic. For $t \in [t_k, t_{k+1}]$ define

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)]$$

(i) Show that $I(t) - I(s)$ is independent of $\mathcal{F}(s)$.

(ii) $I(t) - I(s)$ is normally distributed with mean zero and variance $\int_s^t \Delta^2(u)\mathrm{d}u$

(iii) Show that $I(t), 0 \leq t \leq T$ is a martingale.

(iv) $\mathbb{E}[I^2(t) - \int_0^t \Delta^2(u)\mathrm{d}u|\mathcal{F}(s)] = I^2(s) - \int_0^s \Delta^2(u)\mathrm{d}u$

Proof

We first show that without loss of generality we can assume that $s = t_\ell$ for some $\ell \leq k$. Indeed if $t_k < s < t \leq t_{k+1}$, then since $\Delta(s) = \Delta(t_k)$, it holds that

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(s) - W(t_k)] + \Delta(s)[W(t) - W(s)].$$

Similarly, if $s \in [t_\ell, t_{\ell+1}]$ for $\ell < k$, it then holds that

$$\begin{aligned} I(t) = &\sum_{j=0}^{\ell-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] \\ &+ \Delta(t_\ell)[W(s) - W(t_\ell)] + \Delta(s)[W(t_{\ell+1}) - W(s)] \\ &+ \sum_{j=0}^{\ell+1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)]. \end{aligned}$$

So we may assume that $s = t_\ell$. Therefore,

$$I(s) = \sum_{j=0}^{\ell-1} \Delta(t_j)[W(t_{j+1}) - W(t_\ell)] + \underbrace{\Delta(t_\ell)[W(s) - W(t_\ell)]}_{=0}.$$

Thus,

$$I(t) - I(s) = \sum_{j=\ell}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)]$$

(i) Notice that $W(t_{j+1}) - W(t_j), \ell \leq j \leq k-1$ and $W(t) - W(t_k)$ are independent of $\mathcal{F}(s)$. Since Δ coefficients in the displayed equation for $I(t) - I(s)$ are nonrandom, we conclude that $I(t) - I(s)$ is independent of $\mathcal{F}(s)$.

(ii) Being a sum of normally distributed independent random variables, $I(t) - I(s)$ is normally distributed itself. We have that

$$\begin{aligned}\mathbb{E} I(t) - I(s) &= \mathbb{E} \sum_{j=\ell}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)] \\ &= \sum_{j=\ell}^{k-1} \Delta(t_j)\mathbb{E}[W(t_{j+1}) - W(t_j)] + \Delta(t_k)\mathbb{E}[W(t) - W(t_k)] \\ &= 0.\end{aligned}$$

Moreover,

$$\begin{aligned}\operatorname{Var} I(t) - I(s) &= \sum_{j=\ell}^{k-1} \Delta^2(t_j) \operatorname{Var}[W(t_{j+1}) - W(t_j)] + \Delta^2(t_k) \operatorname{Var}[W(t) - W(t_k)] \\ &= \sum_{j=\ell}^{k-1} \Delta^2(t_j)(t_{j+1} - t_j) + \Delta^2(t_k)(t - t_k) \\ &= \int_s^t \Delta^2(u)\mathrm{d}u.\end{aligned}$$

(iii) Notice that $I(s)$ is $\mathcal{F}(s)$-measurable. Since $I(t) - I(s)$ is independent of $\mathcal{F}(s)$, we have that

$$\begin{aligned}\mathbb{E}[I(t)|\mathcal{F}(s)] &= \mathbb{E}[I(t) - I(s)|\mathcal{F}(s)] + I(s) \\ &= \mathbb{E}[I(t) - I(s)] + I(s) \\ &= I(s).\end{aligned}$$

(iv) We need to show that

$$\mathbb{E}[I^2(t)|\mathcal{F}(s)] = I^2(s) + \int_s^t \Delta^2(u)\mathrm{d}u$$

However,

$$\begin{aligned}\mathbb{E}[I^2(t)|\mathcal{F}(s)] &= \mathbb{E}[(I(t) - I(s) + I(s))^2 \,|\mathcal{F}(s)] \\ &= \mathbb{E}[(I(t) - I(s))^2 \,|\mathcal{F}(s)] + \mathbb{E}[I^2(s)|\mathcal{F}(s)] + 2\mathbb{E}[(I(t) - I(s))\, I(s)|\mathcal{F}(s)] \\ &= \mathbb{E}[(I(t) - I(s))^2] + I^2(s) + 2I(s)\mathbb{E}[I(t) - I(s)] \\ &= I^2(s) + \int_s^t \Delta^2(u)\mathrm{d}u.\end{aligned}$$