Exercise 4.20 (Local time)

Itô formula asserts that

$$\mathrm{d}f(W(t)) = f'(W(t))\mathrm{d}W(t) + \tfrac{1}{2}f''(W(t))\mathrm{d}t$$

where f' is defined and continuous everywhere and furthermore $f''(x)$ is bounded and defined everywhere except finitely many points.

1. Show that if f' is not defined at some point, then Itô formula does not necessarily hold. Hint: $f = (x-K)^+$ is a counter example.

2. To see why this happens define the local time of the Brownian motion at K to be

$$L_K(T) = \lim_{n\to+\infty} n \int_0^T \mathbf{1}_{(K-\frac{1}{2n},K+\frac{1}{2n})}\left(W(t)\right)\mathrm{d}t$$

Show that $L_K(T)$ is never $+\infty$ and it cannot be zero almost surely.

Proof

1. If Itô formula is applicable, then

$$f(W(T)) = \underbrace{f(W(0))}_{=0} + \int_0^T \underbrace{f'(W(t))}_{\mathbf{1}_{\{W(t)>K\}}}\mathrm{d}W(t) + \tfrac{1}{2}\int_0^T \underbrace{f''(W(t))}_{=0}\mathrm{d}t.$$

Here $\int_0^T \mathbf{1}_{\{W(t)>K\}}\mathrm{d}W(t)$ is martingale with expectation zero. Taking expectations from both sides yields

$$\mathbb{E}\left(W(T)-K\right)^+ = 0$$

This in turn implies that $W(T)\le K$ almost surely. But $W(T)$ is normal and so $W(T)>K$ with positive probability.

2. Define

$$f_n(x) = \begin{cases} 0 & \text{if } x \le K-\frac{1}{2n}, \\ \frac{n}{2}(x-K)^2 + \frac{1}{2}(x-K) + \frac{1}{8n} & \text{if } K-\frac{1}{2n}\le x\le K+\frac{1}{2n}, \\ x-K & \text{if } x \ge K+\frac{1}{2n} \end{cases}$$

It is easily verified that

- f_n' is defined and continuous everywhere
- $f_n''(x)$ is bounded and defined everywhere except at two points $x = K\pm\frac{1}{2n}$
- For every $x\in\mathbb{R}$, $\lim_{n\to+\infty} f_n(x) = (x-K)^+$
- Except at $x=K$, $\lim_{n\to+\infty} f_n'(x) = \mathbf{1}_{(K,+\infty)}(x)$

Thus, Itô formula gives

$$f_n(W(T)) = f_n(W(0)) + \int_0^T f_n'(W(t))\mathrm{d}W(t) + \int_0^T f_n''(W(t))\mathrm{d}t$$

Taking the limit $n \to +\infty$, we obtain that

$$(W(T)-K)^+ = \underbrace{(W(0)-K)^+}_{=0} + \int_0^T \mathbf{1}_{(K,+\infty)}(W(t))\mathrm{d}W(t) + \lim_{n\to+\infty} n \int_0^T \mathbf{1}_{(K-\frac{1}{2n},K+\frac{1}{2n})}(W(t))\,\mathrm{d}t$$

Rewriting,

$$L_K(T) = (W(T)-K)^+ - \int_0^T \mathbf{1}_{(K,+\infty)}(W(t))\mathrm{d}W(t)$$

Since $(W(T)-K)^+$ is finite, $L_K(T)$ can never be $+\infty$. Now if $L_K(T) = 0$ almost surely, it then holds Note that

$$0 = \mathbb{E}L_K(T) = \mathbb{E}(W(T)-K)^+ - \mathbb{E}\int_0^T \mathbf{1}_{(K,+\infty)}(W(t))\mathrm{d}W(t).$$

But $\int_0^T \mathbf{1}_{(K,+\infty)}(W(t))\mathrm{d}W(t)$ is an Itô integral and hence has expectation zero. Therefore, $\mathbb{E}(W(T)-K)^+ = 0$ which is a contradiction as explained in item 1.