Exercise 4.21 (Stop-loss start-gain paradox)

Stop-loss start-gain strategy works as follows: Borrow from the money market to buy a share of stock whenever the stock price rises across level K and sell the share, repaying the money market debt, when it falls back across level K. In other words,

$$\Delta(t) = \mathbf{1}_{(K,+\infty)}(S(t))$$

Therefore,

$$X(T) = \sigma \int_0^T \mathbf{1}_{(K,+\infty)}(S(u))S(u)\mathrm{d}W(u)$$

Is the following true:

$$X(T) = (S(T) - K)^+$$

Proof

Define

$$f_n(x) = \begin{cases} 0 & \text{if } x \leq K - \frac{1}{2n}, \\ \frac{n}{2}(x-K)^2 + \frac{1}{2}(x-K) + \frac{1}{8n} & \text{if } K - \frac{1}{2n} \leq x \leq K + \frac{1}{2n}, \\ x - K & \text{if } x \geq K + \frac{1}{2n} \end{cases}$$

It is easily verified that

- f_n' is defined and continuous everywhere
- $f_n''(x)$ is bounded and defined everywhere except at two points $x = K \pm \frac{1}{2n}$
- For every $x \in \mathbb{R}$, $f_n'(x) \geq 0$.
- For every $x \in \mathbb{R}$, $\lim_{n\to+\infty} f_n(x) = (x-K)^+$
- Except at $x = K$, $\lim_{n\to+\infty} f_n'(x) = \mathbf{1}_{(K,+\infty)}(x)$

Itô formula gives

$$\begin{aligned} \mathrm{d}f_n(S(t)) &= f_n'(S(t))\mathrm{d}S(t) + \tfrac{1}{2}f_n''(S(t))\mathrm{d}S(t)\mathrm{d}S(t) \\ &= \sigma f_n'(S(t))S(t)\mathrm{d}W(t) + \tfrac{n}{2}\mathbf{1}_{|S(t)-K|<\frac{1}{2n}}\sigma^2 S^2(t)\mathrm{d}t. \end{aligned}$$

Thus,

$$f_n(S(t)) = \sigma \int_0^t f_n'(S(u))S(u)\mathrm{d}W(u) + \tfrac{n\sigma^2}{2}\int_0^t \mathbf{1}_{|S(u)-K|<\frac{1}{2n}} S^2(u)\mathrm{d}u$$

Since $f_n'(x) \geq 0$, taking limit $n \to +\infty$ gives

$$(S(t)-K)^+ = \sigma \int_0^t \mathbf{1}_{(K,+\infty)}(S(u))S(u)\mathrm{d}W(u) + \lim_{n\to+\infty} \tfrac{n\sigma^2}{2}\int_0^t \mathbf{1}_{|S(u)-K|<\frac{1}{2n}} S^2(u)\mathrm{d}u$$

Integration gives

$$\mathbb{E}(S(T)-K)^+ = \tfrac{\sigma^2}{2}\mathbb{E}\lim_{n\to+\infty} n \int_0^T \mathbf{1}_{|S(u)-K|<\frac{1}{2n}} S^2(u)\mathrm{d}u$$

We claim that $\mathbb{E}(S(T)-K)^+ > 0$. In fact, otherwise, we should have that $S(T) \leq K$ almost surely. This is untrue as

$$\begin{aligned} S(T) \leq K \iff & S(0)\exp\left(\sigma W(T) - \tfrac{\sigma^2 T}{2}\right) \leq K \\ \iff & W(T) \leq \tfrac{1}{\sigma}\log\frac{K}{S(0)} + \tfrac{\sigma T}{2} \end{aligned}$$

However, $W(T)$ is normally distributed with variance T. This means that the above inequality is violated with positive probability. In conclusion,

$$\lim_{n\to+\infty} \tfrac{n\sigma^2}{2}\int_0^t \mathbf{1}_{|S(u)-K|<\frac{1}{2n}} S^2(u)\mathrm{d}u > 0 \quad \text{with positive probability!}$$

In other words,

$$(S(T)-K)^+ > X(T) = \sigma\int_0^T \mathbf{1}_{(K,+\infty)}(S(u))S(u)\mathrm{d}W(u) \quad \text{with positive probability!}$$