Exercise 4.3

Consider the following partition of $[0, T]$

$$t_0 = 0 < t_1 = s < t_2 = t = T$$

Define a simple random function Δ as below

$$\Delta(u) = \begin{cases} \Delta(0) & \text{if } u < s, \\ W(s) & \text{if } s \le u \le T. \end{cases}$$

Which of the following assertions are true?

(i) $I(t) - I(s)$ is independent of $\mathcal{F}(s)$

(ii) $I(t) - I(s)$ is normally distributed

(iii) $\mathbb{E}[I(t)|\mathcal{F}(s)] = I(s)$

(iv) $\mathbb{E}[I^2(t) - \int_0^t \Delta^2(u)\mathrm{d}u|\mathcal{F}(s)] = I^2(s) - \int_0^s \Delta^2(u)\mathrm{d}u$

Proof

Notice that

$$I(t) = \Delta(0)W(s) + W(s)\left(W(t) - W(s)\right), \quad I(s) = \Delta(0)W(s).$$

Denote

$$D := I(t) - I(s) = W(s)\left(W(t) - W(s)\right).$$

(i) Since $W(t) - W(s)$ is independent of $\mathcal{F}(s)$, then D being independent of $\mathcal{F}(s)$ means that $W(s) = D\left(W(t) - W(s)\right)^{-1}$ is independent of $\mathcal{F}(s)$. But we know that $W(s)$ is $\mathcal{F}(s)$-measurable and of course is not a constant. Thus, D is not $\mathcal{F}(s)$-independent.

(ii) Notice that $W(s)$ and $W(t) - W(s)$ are independent. Therefore, since Kurtosis for a normal random variable is 3, the following holds

$$\mathbb{E}D^4 = \mathbb{E}W^4(s)\mathbb{E}\left(W(t) - W(s)\right)^4 = 9s^2(t-s)^2$$

Moreover,

$$\mathbb{E}D^2 = \mathbb{E}W^2(s)\mathbb{E}\left(W(t) - W(s)\right)^2 = s(t-s).$$

Therefore, $\mathbb{E}D^4 \ne 3\left(\mathbb{E}D^2\right)^2$ and consequently D cannot be normally distributed.

(iii) $\mathbb{E}[D|\mathcal{F}(s)] = W(s)\mathbb{E}[W(t) - W(s)|\mathcal{F}(s)] = 0.$

(iv) $\int_0^t \Delta^2(u)\mathrm{d}u = \Delta^2(0)s + (t-s)W^2(s)$ is $\mathcal{F}(s)$-measurable. We need to verify whether

$$\mathbb{E}[I^2(t)|\mathcal{F}(s)] = I^2(s) + \int_s^t \Delta^2(u)\mathrm{d}u$$

Continuing,

$$\begin{aligned}
\mathbb{E}[I^2(t)|\mathcal{F}(s)] &= \mathbb{E}[(I(s)+D)^2\,|\mathcal{F}(s)] \\
&= \mathbb{E}[I^2(s)+2DI(s)+D^2|\mathcal{F}(s)] \\
&= I^2(s)+2I(s)\mathbb{E}[D|\mathcal{F}(s)]+\mathbb{E}[D^2|\mathcal{F}(s)] \\
&= I^2(s)+\mathbb{E}[D^2|\mathcal{F}(s)] \\
&= I^2(s)+W^2(s)\mathbb{E}[(W(t)-W(s))^2\,|\mathcal{F}(s)] \\
&= I^2(s)+(t-s)W^2(s) \\
&= I^2(s)+\int_s^t \Delta^2(u)\mathrm{d}u.
\end{aligned}$$