Exercise 4.4 (Stratonovich integral))

Let $W(t), t \geq 0$ be a Brownian motion. Fix $T > 0$ and consider $\Pi = \{t_0, \cdots, t_n\}$ to be a partition of $[0, T]$. Define half-sample quadratic variation to be

$$Q_{\frac{\Pi}{2}} = \sum_{j=0}^{n-1} \left(W(t_j^*) - W(t_j)\right)^2$$

Define Stratonovich integral of $W(t)$ w.r.t. $W(t)$ to be

$$\int_0^T W(t) \circ \mathrm{d}W(t) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} W(t_j^*) \left(W(t_{j+1}) - W(t_j)\right)$$

(i) Show that $Q_{\frac{\Pi}{2}} \to \frac{T}{2}$ as $\|\Pi\| \to 0$.

(ii) Show that $\int_0^T W(t) \circ \mathrm{d}W(t) = \frac{W^2(T)}{2}$.

Proof

(i) $W(t_j^*) - W(t_j)$ is normal with mean zero and variance $t_j^* - t_j$. Therefore,

$$\mathbb{E} \sum_{j=0}^{n-1} \left(W(t_j^*) - W(t_j)\right)^2 = \sum_{j=0}^{n-1} t_j^* - t_j = \sum_{j=0}^{n-1} \frac{t_{j+1} - t_j}{2} = \frac{T}{2}.$$

On the other hand,

$$\begin{aligned}
\mathrm{Var} \sum_{j=0}^{n-1} \left(W(t_j^*) - W(t_j)\right)^2 &= \sum_{j=0}^{n-1} \mathrm{Var} \left(W(t_j^*) - W(t_j)\right)^2 \\
&= 3 \sum_{j=0}^{n-1} \mathbb{E} \left(W(t_j^*) - W(t_j)\right)^4 \\
&= 3 \sum_{j=0}^{n-1} \left(t_j^* - t_j\right)^2 \\
&= \frac{3}{4} \sum_{j=0}^{n-1} (t_{j+1} - t_j)^2 \\
&\leq \frac{3\|\Pi\|}{4} \sum_{j=0}^{n-1} t_{j+1} - t_j \\
&= \frac{3\|\Pi\| T}{4} \to 0 \text{ when } \|\Pi\| \to 0.
\end{aligned}$$

(ii) We have that

$$\begin{aligned}
\lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1} W(t_j^*)\left(W(t_{j+1})-W(t_j)\right) &= \lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1} W(t_j)\left(W(t_{j+1})-W(t_j)\right) \\
&\quad + \lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1}\left(W(t_j^*)-W(t_j)\right)\left(W(t_{j+1})-W(t_j^*)+W(t_j^*)-W(t_j)\right) \\
&= \int_0^T W(t)\mathrm{d}W(t) + Q_{\frac{\Pi}{2}} + \lim_{\|\Pi\|\to 0}\textstyle\sum_{j=0}^{n-1}\left(W(t_j^*)-W(t_j)\right)\left(W(t_{j+1})-W(t_j^*)\right) \\
&= \frac{W^2(T)}{2} + \lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1}\left(W(t_j^*)-W(t_j)\right)\left(W(t_{j+1})-W(t_j^*)\right).
\end{aligned}$$

Next,

$$\mathbb{E}\sum_{j=0}^{n-1}\left(W(t_j^*)-W(t_j)\right)\left(W(t_{j+1})-W(t_j^*)\right) = 0.$$

Moreover,

$$\begin{aligned}
\operatorname{Var}\sum_{j=0}^{n-1}\left(W(t_j^*)-W(t_j)\right)\left(W(t_{j+1})-W(t_j^*)\right) &= \sum_{j=0}^{n-1}\operatorname{Var}\left(W(t_j^*)-W(t_j)\right)\left(W(t_{j+1})-W(t_j^*)\right) \\
&= \sum_{j=0}^{n-1}\mathbb{E}\left(W(t_j^*)-W(t_j)\right)^2\left(W(t_{j+1})-W(t_j^*)\right)^2 \\
&= \sum_{j=0}^{n-1}\mathbb{E}\left(W(t_j^*)-W(t_j)\right)^2\mathbb{E}\left(W(t_{j+1})-W(t_j^*)\right)^2 \\
&= \sum_{j=0}^{n-1}\left(t_j^*-t_j\right)\left(t_{j+1}-t_j^*\right) \\
&= \tfrac{1}{4}\sum_{j=0}^{n-1}\left(t_{j+1}-t_j\right)^2 \\
&\le \frac{\|\Pi\|}{4}\sum_{j=0}^{n-1} t_{j+1}-t_j \\
&= \frac{\|\Pi\|T}{4}.
\end{aligned}$$

Therefore,

$$\lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1}\left(W(t_j^*)-W(t_j)\right)\left(W(t_{j+1})-W(t_j^*)\right) = 0.$$