Exercise 4.5 (Solving generalized GBM equation)

Let

$$X(t) = \int_0^t \sigma(s)\mathrm{d}W(s) + \int_0^t \left(\alpha(s) - \tfrac{1}{2}\sigma^2(s)\right)\mathrm{d}s.$$

Define the generalized geometric Brownian motion as below

$$S(t) = S(0)e^{X(t)} \tag{1}$$

Itô formula yields that

$$\mathrm{d}S(t) = \alpha(t)S(t)\mathrm{d}t + \sigma(t)S(t)\mathrm{d}W(t)$$

In this exercise, we show the reverse direction. Namely, the solution to this differential equation must be Eq (1).

Proof

Let $f(x) = \log x$. Then

$$\begin{aligned}\mathrm{d}f(S(t)) &= \frac{\mathrm{d}S(t)}{S(t)} - \frac{\mathrm{d}S(t)\mathrm{d}S(t)}{S^2(t)} \\ &= \alpha(t)\mathrm{d}t + \sigma(t)\mathrm{d}W(t) - \sigma^2(t)\mathrm{d}t \\ &= \sigma(t)\mathrm{d}W(t) + \left(\alpha(t) - \sigma^2(t)\right)\mathrm{d}t.\end{aligned}$$

Integrating both side gives that

$$f(S(T)) = f(S(0)) + X(t)$$

Exponentiation of both sides completes the proof.