

Exercise 4.6

Let $S(t) = S(0) \exp((\alpha - \frac{1}{2}\sigma^2)t + \sigma W(t))$. For $p > 0$, compute $dS^p(t)$.

Proof

Note that

$$S^p(t) = S(0)^p \cdot \exp\left(\underbrace{p(\alpha - \frac{1}{2}\sigma^2)t + p\sigma W(t)}_{:=X(t)}\right).$$

Let $f(x) = S(0)^p e^x$. Itô's lemma then implies

$$\begin{aligned} df(X(t)) &= f'(X(t))dX(t) + \frac{1}{2}f''(X(t))dX(t)dX(t) \\ &= f(X(t)) \cdot [p(\alpha - \frac{1}{2}\sigma^2)dt + p\sigma dW(t) + \frac{1}{2}p\sigma^2 dt] \\ &= pS^p(t) \cdot [\alpha dt + \sigma dW(t)]. \end{aligned}$$