Exercise 4.7

(i) Compute $dW^4(t)$ and expand $W^4(T)$ in terms of an Itô integral and an ordinary one.

(ii) Use the fact that $\mathbb{E}W^2(t) = t$ to show that $\mathbb{E}W^4(T) = 3T^2$.

(iii) Similarly drive a formula for $\mathbb{E}W^6(T)$.

Proof

(i) Recall that

$$\mathrm{d}f(W(t)) = f'(W(t))\mathrm{d}W(t) + \tfrac{1}{2}f''(W(t))\mathrm{d}W(t)\mathrm{d}t.$$

Considering $f(x) = x^4$, we obtain that

$$\mathrm{d}tW^4(t) = 4W^3(t)\mathrm{d}W(t) + 6W^2(t)\mathrm{d}t.$$

Integration gives

$$W^4(T) = 4\int_0^T W^3(t)\mathrm{d}W(t) + 6\int_0^T W^2(t)\mathrm{d}t.$$

(ii) Itô integrals have integral equal to zero. Therefore,

$$\mathbb{E}W^4(T) = 6\int_0^T \mathbb{E}W^2(t)\mathrm{d}t = 6\int_0^T t\mathrm{d}t = 3T^2.$$

(iii) Similar as above, it holds that

$$W^6(T) = 6\int_0^T W^5(t)\mathrm{d}W(t) + 15\int_0^T W^4(t)\mathrm{d}t.$$

Thus,

$$\mathbb{E}W^6(T) = 15\int_0^T \mathbb{E}W^4(t)\mathrm{d}t = 15\int_0^T 3t^2\mathrm{d}t = 15T^3.$$