Exercise 4.8 (Solving the Vasicek equation)

Recall the Vasicek interest rate stochastic differential equation:

$$dR(t) = (\alpha - \beta R(t))dt + \sigma dW(t).$$

Show that

$$R(t) = e^{-\beta t} \cdot R(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \sigma e^{-\beta t} \int_0^t e^{\beta s} dW(s)$$

Proof

Rewriting the differential equation to get the integral form, we have

$$\begin{aligned} R(t) &= \int_0^t (\alpha - \beta R(u))du + \sigma \cdot W(t) \\ &= \alpha t - \beta \int_0^t R(u)du + \sigma \cdot W(t) \end{aligned}$$

Thus,

$$R(t) + \beta \int_0^t R(u)du = \alpha t + \sigma \cdot W(t).$$

On the other hand, Itô product rule gives the following

$$\begin{aligned} de^{\beta t} \cdot R(t) &= e^{\beta t} dR(t) + \beta e^{\beta t} R(t)dt + \beta e^{\beta t} dR(t) \cdot dt \\ &= e^{\beta t} dR(t) + \beta e^{\beta t} R(t)dt \\ &= \alpha e^{\beta t} dt + \sigma e^{\beta t} dW(t). \end{aligned}$$

Integrating both sides, the result immediately follows:

$$e^{\beta t} \cdot R(t) - R(0) = \frac{\alpha}{\beta} \cdot \left(e^{\beta t} - 1\right) + \sigma \int_0^t e^{\beta s} dW(s).$$