## Exercise 4.9

Consider a European call option and define

- $\tau$ : Time to expiry *i.e.*, $T - t$
- $T$ : Option's expiry
- $K$ : Strike price
- $x$ : Time-$t$ stock price
- $c(t, x)$ : Time-$t$ Black-Scholes-Merton price

Then

$$c(t, x) = xN(d_+(\tau, x)) - Ke^{-r\tau}N(d_-(\tau, x))$$

where

$$d_+(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r + \tfrac{1}{2}\sigma^2\right)\tau\right], \quad d_-(\tau, x) = d_+(\tau, x) - \sigma\sqrt{\tau}$$

In this exercise, we show that $c$ satisfies the following items

- Black-Scholes-Merton PDE:

$$c_t(t, x) + rxc_x(t, x) + \tfrac{1}{2}\sigma^2x^2c_{xx}(t, x) = rc(t, x) \text{ where } t \in [0, T), \quad x > 0.$$

- Terminal condition:

$$\lim_{t \uparrow T} c(t, x) = (x - K)^+ \text{ where } x > 0, x \neq K.$$

- Boundary conditions:

$$\lim_{x \downarrow 0} c(t, x) = 0, \lim_{x \to +\infty}\left[c(t, x) - \left(x - e^{-r(T-t)}K\right)\right] = 0 \text{ where } t \in [0, T).$$

1. Prove that

$$Ke^{-r(T-t)}N'(d_-) = xN'(d_+).$$

2. Show that

$$\underbrace{c_x}_{\text{delta}} = N(d_+) \text{ and } \underbrace{c_t}_{\text{theta}} = -rKe^{-r\tau}N(d_-) - \frac{\sigma x}{2\sqrt{\tau}}N'(d_+)$$

3. Complete the proof!

### Proof

1. Recall that

$$N(y) = \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{z^2}{2}}\,\mathrm{d}z.$$

Therefore, $N'(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$. Note that

$$Ke^{-r(T-t)}N'(d_-) = xN'(d_+) \iff \tfrac{1}{2}\left(d_+^2 - d_-^2\right) = \log\frac{x}{K} + r(T - t)$$

However,

$$\begin{aligned}
\tfrac{1}{2}\left(d_+^2 - d_-^2\right) &= \tfrac{1}{2}\left(d_+ - d_-\right)\left(d_+ + d_-\right) \\
&= \tfrac{1}{2}\sigma\sqrt{\tau}\cdot\left(\frac{2}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r + \tfrac{1}{2}\sigma^2\right)\tau\right] - \sigma\sqrt{\tau}\right) \\
&= \log\frac{x}{K} + r\tau
\end{aligned}$$

2. To compute delta, we have that

$$\begin{aligned}
c_x &= N'(d_+) + N(d_+) - \frac{Ke^{-r\tau}}{x}N'(d_-(\tau,x)) \\
&= N(d_+) + x^{-1}\left[xN'(d_+) - Ke^{-r(T-t)}N'(d_-)\right] \\
&= N(d_+)
\end{aligned}$$

To compute theta, note that

$$c_t = -c_\tau = -xN'(d_+)d_+'(\tau,x) + rKe^{-r\tau}N(d_-) - Ke^{-r\tau}N'(d_-)d_-'(\tau,x)$$

It remains to show that $xN'(d_+)d_+' - Ke^{-r\tau}N'(d_-)d_-' = \frac{\sigma x}{2\sqrt{\tau}}N'(d_+)$. However,

$$\begin{aligned}
xN'(d_+)d_+' - Ke^{-r\tau}N'(d_-)d_-' &= xN'(d_+)d_+' - xN'(d_+)d_-' \\
&= xN'(d_+)\cdot\left(d_+' - d_-'\right) \\
&= \frac{\sigma x}{2\sqrt{\tau}}N'(d_+)
\end{aligned}$$

3. We have that

$$\begin{aligned}
c_t(t,x) + rxc_x(t,x) + \tfrac{1}{2}\sigma^2x^2c_{xx}(t,x) &= -rKe^{-r\tau}N(d_-) - \frac{\sigma x}{2\sqrt{\tau}}N'(d_+) \\
&\quad + rxN(d_+) + \frac{1}{2}\sigma^2x^2N'(d_+)\cdot\frac{1}{x\sigma\sqrt{\tau}} \\
&= -rKe^{-r\tau}N(d_-) + rxN(d_+) - \\
&= r\left[xN(d_+) - Ke^{-r\tau}N(d_-)\right] \\
&= rc(t,x).
\end{aligned}$$

Next,

$$N(d_+) = N(d_-) = \begin{cases} 0 & x = K \\ N(+\infty) = 1 & x > K \\ N(-\infty) = 0 & x < K. \end{cases}$$

Therefore,

$$\lim_{t\uparrow T} c(t,x) = \begin{cases} 0 & x \le K \\ x - K & x > K \end{cases} = (x-K)^+.$$

Next,

$$\lim_{x\downarrow 0} N(d_+) = \lim_{x\downarrow 0} N(d_-) = 0.$$

Thus,

$$\lim_{x\downarrow 0} c(t,x) = 0.$$

Finally,

$$c(t,x) - \left(x - e^{-r(T-t)}K\right) = x\left(N(d_+) - 1\right) - e^{-r(T-t)}K\left(N(d_-) - 1\right)$$

Since

$$\lim_{x\uparrow +\infty} N(d_+) = \lim_{x\downarrow 0} N(d_-) = 1,$$

it holds that

$$\begin{aligned}
\lim_{x\uparrow +\infty} c(t,x) - \left(x - e^{-r(T-t)}K\right) &= \lim_{x\uparrow +\infty} x\left(N(d_+) - 1\right) \\
&= \lim_{x\uparrow +\infty} -xN(-d_+) \\
&= -\lim_{x\uparrow +\infty} \frac{N(-d_+)}{x^{-1}} \\
&= \tfrac{1}{\sqrt{2\pi}} \lim_{x\uparrow +\infty} \frac{e^{-\frac{d_+^2}{2}}}{x^{-2}}
\end{aligned}$$

On the other hand, for any fixed $c > 0$, for $x$ large enough, $d_+ \geq c$. Therefore, for $x$ large enough, it holds that

$$\frac{d_+^2}{2} \geq c \log x.$$

Therefore, fixing any $c > 2$,

$$\begin{aligned}
\lim_{x\uparrow +\infty} \frac{e^{-\frac{d_+^2}{2}}}{x^{-2}} &= \lim_{x\uparrow +\infty} \frac{e^{-c\log x}}{x^{-2}} \\
&\leq \lim_{x\uparrow +\infty} \frac{x^{-c}}{x^{-2}} \\
&= \lim_{x\uparrow +\infty} x^{-c+2} \\
&= 0.
\end{aligned}$$