Exercise 5.1

Consider the stock price process as below

$$S(t) = S(0) \cdot \exp\left(\int_0^t \sigma(s) dW(s) + \int_0^t \left(\alpha(s) - \tfrac{1}{2} \cdot \sigma^2(s) \right) ds \right)$$

Also, denote

$$D(t) = e^{-\int_0^t R(s) ds}$$

Establish the following derivation in two different methods. First using Itô formula and second Itô product rule.

$$d(D(t)S(t)) = \sigma(t)D(t)S(t) \cdot [\Theta(t)dt + dW(t)]$$

where $\Theta(t) = \frac{\alpha(t)-R(t)}{\sigma(t)}$ is the market price of risk.

Proof

(1) Note that

$$\begin{aligned} D(t)S(t) &= S(0) \cdot \exp\left(\int_0^t \sigma(s) dW(s) + \int_0^t \left(\alpha(s) - R(s) - \tfrac{1}{2} \cdot \sigma^2(s) \right) ds \right) \\ &= S(0) \cdot e^{X(t)} \end{aligned}$$

where $X(t) = \int_0^t \sigma(s) dW(s) + \int_0^t \left(\alpha(s) - R(s) - \frac{1}{2} \cdot \sigma^2(s) \right) ds$. Let $f(x) := S(0) \cdot e^x$. Itô formula then gives

$$df(X(t)) = f'(X(t))dX(t) + \tfrac{1}{2} \cdot f''(X(t))dX(t)dX(t).$$

On the other hands,

$$dX(t) = \sigma(t)dW(t) + \left(\alpha(t) - R(t) - \tfrac{1}{2} \cdot \sigma^2(t) \right) dt.$$

Putting pieces together,

$$\begin{aligned} \mathrm{d}f(X(t)) &= S(0) \cdot e^{X(t)} \cdot \sigma(t)\mathrm{d}W(t) + S(0) \cdot e^{X(t)} \cdot \left(\alpha(t) - R(t) - \tfrac{1}{2} \cdot \sigma^2(t) \right) \mathrm{d}t \\ &\quad + S(0) \cdot e^{X(t)} \cdot \frac{1}{2} \cdot \sigma^2(t)\mathrm{d}t \\ &= f(X(t)) \cdot [\sigma(t)\mathrm{d}W(t) + (\alpha(t) - R(t))\, \mathrm{d}t] \\ &= f(X(t)) \cdot [\sigma(t)\mathrm{d}W(t) + \Theta(t)\mathrm{d}t] \end{aligned}$$

where $\Theta(t) = \frac{\alpha(t)-R(t)}{\sigma(t)}$ is the market price of risk.

(2) Using Itô product rule, we derive

$$d(D(t)S(t)) = D(t)dS(t) + S(t)dD(t) + dD(t)dS(t)$$

Next

$$\mathrm{d}D(t) = -R(t)D(t)\mathrm{d}t \text{ and } \mathrm{d}S(t) = S(t) \cdot (\alpha(t)\mathrm{d}t + \sigma(t)\mathrm{d}W(t)$$

Continuing, we have that

$$d(D(t)S(t)) = D(t)S(t) \cdot [\alpha(t)\mathrm{d}t + \sigma(t)dW(t) - R(t)\mathrm{d}t]$$

We immediately derive the desired result.