Exercise 5.10 (Chooser option)

Consider a date t_0 between 0 and T. A chooser option gives its owner the right at time t_0 to have the possession of either a call or put option with the strike price K. Denote by $C(t), P(t)$ and $F(t)$ the call option, put option and future price at time t expiring at T; all with strike price K.

1. Show that at time t_0 the value of the chooser option is

$$C(t_0) + \max\{0, -F(t_0)\} = C(t_0) + \left(e^{-r(T-t_0)}K - S(t_0)\right)^+$$

2. Show that at time 0 a chooser option is worth the sum of values of the following two securities:
   - A call expiring at time T with strike price K
   - A put expiring at time t_0 with strike price $e^{-r(T-t_0)}K$

Proof

1. Put-call parity states that

$$C(t) - P(t) = F(t)$$

   Therefore,

$$C(t_0) + \max\{0, -F(t_0)\} = \begin{cases} C(t_0) & \text{if } F(t_0) > 0 \\ P(t_0) & \text{if } F(t_0) < 0. \end{cases}$$

   At time t_0, we choose call over put if and only if $C(t_0) > P(t_0)$. However, using put-call parity, first item follows since

$$C(t_0) > P(t_0) \iff F(t_0) > 0.$$

2. To compute the chooser option's value at time 0 (denoted by ν), we will discount its value at time t_0 as follows:

$$\nu := \tilde{\mathbb{E}}\left[e^{-rt_0}\left(C(t_0) + \left(e^{-r(T-t_0)}K - S(t_0)\right)^+\right)\right]$$

   Recall that

$$C(t) = \tilde{\mathbb{E}}\left[e^{-r(T-t)}\left(S(T) - K\right)^+ |\mathcal{F}(t)\right], P(t) = \tilde{\mathbb{E}}\left[e^{-r(T-t)}\left(K - S(T)\right)^+ |\mathcal{F}(t)\right].$$

   Iterated conditioning yields

$$\begin{aligned} \nu &= \tilde{\mathbb{E}}\left[e^{-rt_0} \cdot C(t_0) + e^{-rt_0}\left(e^{-r(T-t_0)}K - S(t_0)\right)^+\right] \\ &= \tilde{\mathbb{E}}\left[e^{-rt_0} \cdot e^{-r(T-t_0)}\left(S(T) - K\right)^+\right] + \tilde{\mathbb{E}}\left[e^{-rt_0}\left(e^{-r(T-t_0)}K - S(t_0)\right)^+\right] \\ &= \underbrace{\tilde{\mathbb{E}}\left[e^{-rT}\left(S(T) - K\right)^+\right]}_{\text{Call, expiry } = T\text{, strike } = K} + \underbrace{\tilde{\mathbb{E}}\left[e^{-rt_0}\left(e^{-r(T-t_0)}K - S(t_0)\right)^+\right]}_{\text{Put, expiry } = t_0\text{, strike } = e^{-r(T-t_0)}K} \end{aligned}$$