Exercise 5.11 (Hedging a cash flow)

Let $W(t), 0 \leq t \leq T$ be a Brownian motion ad let $\mathcal{F}(t)$ be the resulting filtration. Denote $\alpha(t), R(t)$ and $\sigma(t)$ to be mean rate of return, interest rate and volatility respectively. Consider the following stock price

$$\mathrm{d}S(t) = \alpha(t)S(t)\mathrm{d}t + \sigma(t)S(t)\mathrm{d}W(t), \quad 0 \leq t \leq T.$$

Suppose that an agent pays a cash flow $C(t)$ at each time t where $C(t)$ is an a.p. If she holds $\Delta(t)$ shares of stock at each time t, differential of her portfolio will be

$$\mathrm{d}X(t) = \Delta(t)\mathrm{d}S(t) + R(t)\left(X(t) - \Delta(t)S(t)\right)\mathrm{d}t - C(t)\mathrm{d}t \tag{1}$$

Show that there exists a non-random value $X(0)$ and a portfolio process $\Delta(t), 0 \leq t \leq T$, such that $X(T) = 0$ a.s. Hint: Utilize the following process

$$\tilde{M}(t) = \tilde{\mathbb{E}}\left[\int_0^T D(u)C(u)\mathrm{d}u | \mathcal{F}(t)\right], \quad 0 \leq t \leq T.$$

Here $D(t)$ is the discount process.

Proof

We begin by noting that there exists an a.p. $\tilde{\Gamma}(t), 0 \leq t \leq T$ such that

$$\tilde{M}(t) = \tilde{M}(0) + \int_0^t \tilde{\Gamma}(u)\mathrm{d}\tilde{W}(u).$$

Here

$$\tilde{M}(0) = \tilde{\mathbb{E}}\left[\int_0^T D(u)C(u)\mathrm{d}u\right] \text{ and } \tilde{W}(t) := W(t) + \int_0^t \Theta(u)\mathrm{d}u \quad \text{with } \Theta(u) := \frac{\alpha(u) - R(u)}{\sigma(u)}.$$

Recall that

$$\mathrm{d}(D(t)S(t)) = \sigma(t)D(t)S(t)\mathrm{d}\tilde{W}(t).$$

Multiplying both sides of (1) with $D(t)$ and re-grouping yields

$$\begin{aligned}
\mathrm{d}(D(t)X(t)) &= \Delta(t) \cdot \mathrm{d}(D(t)S(t)) - D(t)C(t)\mathrm{d}t \\
&= \Delta(t)\sigma(t)D(t)S(t)\mathrm{d}\tilde{W}(t) - D(t)C(t)\mathrm{d}t
\end{aligned}$$

Integrating both sides, we obtain that

$$D(T)X(T) = X(0) + \int_0^T \Delta(u)\sigma(u)D(u)S(u)\mathrm{d}\tilde{W}(u) - \int_0^T D(u)C(u)\mathrm{d}u$$

Next, notice that

$$\tilde{M}(T) = \int_0^T D(u)C(u)\mathrm{d}u = \tilde{M}(0) + \int_0^T \tilde{\Gamma}(u)\mathrm{d}\tilde{W}(u).$$

We conclude that

$$D(T)X(T) = X(0) + \int_0^T \Delta(u)\sigma(u)D(u)S(u)\mathrm{d}\tilde{W}(u) - \int_0^T \tilde{\Gamma}(u)\mathrm{d}\tilde{W}(u) - \tilde{M}(0).$$

Putting pieces together, we conclude that letting

$$X(0) = \tilde{\mathbb{E}}\left[\int_0^T D(u)C(u)\mathrm{d}u\right] \text{ and } \Delta(u) := \frac{\tilde{\Gamma}(u)}{\sigma(u)D(u)S(u)}$$

yields that $X(T) = 0$ a.s.