Exercise 5.12 (Correlation under change of measure)

Multidimensional market model for m stocks is defined as follows

$$\mathrm{d}S_i(t) = \alpha_i(t)S_i(t) + S_i(t)\sum_{j=1}^{d}\sigma_{ij}(t)\mathrm{d}W_j(t), \quad \forall i \in [1, m]$$

Let $\sigma_i^2(t) := \sum_{j=1}^{d}\sigma_{ij}^2(t)$ and define

$$B_i(t) := \sum_{j=1}^{d}\int_0^t \frac{\sigma_{ij}(u)}{\sigma_i(u)}\mathrm{d}W_j(u)$$

Market price of risk equations are defined as below

$$\alpha_i(t) - R(t) = \sum_{j=1}^{d}\sigma_{ij}(t)\Theta_j(t), \quad \forall i \in [1, m]$$

Assume that these system of equations have a solution $\Theta_1(t), \cdots, \Theta_d(t)$. Denote by $\tilde{\mathbb{P}}$ the resulting risk-neutral measure under which the following processes are independent Brownian motions

$$\tilde{W}_j(t) = W_j(t) + \int_0^t \Theta_j(u)\mathrm{d}u, \quad \forall j \in [1, d].$$

Define

$$\gamma_i(t) = \sum_{j=1}^{d}\frac{\sigma_{ij}(t)\Theta_j(t)}{\sigma_i(t)}, \quad \forall i \in [1, m].$$

1. Show that $\tilde{B}_i(t) = B_i(t) + \int_0^t \gamma_i(u)\mathrm{d}u$ is a Brownian motion under $\tilde{\mathbb{P}}$.
2. Show that
$$\mathrm{d}\tilde{S}_i(t) = R(t)\tilde{S}_i(t)\mathrm{d}t + \sigma_i(t)\tilde{S}_i(t)\mathrm{d}\tilde{W}_i(t)$$
3. Show that
$$\mathrm{d}B_i(t)\mathrm{d}B_k(t) = \rho_{ik}(t)\mathrm{d}t \Rightarrow \mathrm{d}\tilde{B}_i(t)\mathrm{d}\tilde{B}_k(t) = \rho_{ik}(t)\mathrm{d}t.$$
4. Show that if $\rho_{ik}(t)$ is not random then
$$\tilde{\mathbb{E}}\left[\tilde{B}_i(t)\tilde{B}_k(t)\right] = \mathbb{E}\left[B_i(t)B_k(t)\right] = \int_0^t \rho_{ik}(s)\mathrm{d}s.$$
5. Show that $\tilde{\mathbb{E}}\left[\tilde{B}_i(t)\tilde{B}_k(t)\right] = \mathbb{E}\left[B_i(t)B_k(t)\right]$ does not necessarily hold if $\rho_{ik}(t)$ is random.

Proof

1. Note that $\tilde{B}_i(0) = 0$ and $\tilde{B}_i(t)$ has continuous path. Indeed, $\int_0^t \gamma_i(u)\mathrm{d}u$ is continuous as function of t as γ_i is integrable and thus bounded around some neighborhood of u, for all u. We have that

$$\begin{aligned}
\mathrm{d}\tilde{B}_i(t) &= \sum_{j=1}^{d} \frac{\sigma_{ij}(t)}{\sigma_i(t)}\mathrm{d}W_j(t) + \gamma_i(t)\mathrm{d}t \\
&= \sum_{j=1}^{d} \frac{\sigma_{ij}(t)}{\sigma_i(t)}\mathrm{d}W_j(t) + \sum_{j=1}^{d} \frac{\sigma_{ij}(t)\Theta_j(t)}{\sigma_i(t)}\mathrm{d}t \\
&= \sum_{j=1}^{d} \frac{\sigma_{ij}(t)}{\sigma_i(t)}\left(\mathrm{d}W_j(t) + \Theta_j(t)\mathrm{d}t\right) \\
&= \sum_{j=1}^{d} \frac{\sigma_{ij}(t)}{\sigma_i(t)}\mathrm{d}\tilde{W}_j(t).
\end{aligned}$$

Therefore, B_i is a martingale as it is expressed as sum of Itô integrals as below

$$\tilde{B}_i(t) = \sum_{j=1}^{d} \int_0^t \frac{\sigma_{ij}(u)}{\sigma_i(u)}\mathrm{d}\tilde{W}_j(u).$$

Continuing,

$$\mathrm{d}\tilde{B}_i(t)\mathrm{d}\tilde{B}_i(t) = \sum_{j=1}^{d} \frac{\sigma_{ij}^2(t)}{\sigma_i^2(t)}\mathrm{d}\tilde{W}_j(t)\mathrm{d}\tilde{W}_j(t) = \sum_{j=1}^{d} \frac{\sigma_{ij}^2(t)}{\sigma_i^2(t)}\mathrm{d}t = \mathrm{d}t.$$

Levy's theorem implies that $\tilde{B}_i(t)$ is $\tilde{\mathbb{P}}$-martingale.

2. Note that

$$\begin{aligned}
\mathrm{d}S_i(t) &= \alpha_i(t)S_i(t)\mathrm{d}t + \sigma_i(t)S_i(t)\mathrm{d}B_i(t) \\
&= \left(R(t) + \sum_{j=1}^{d} \sigma_{ij}(t)\Theta_j(t)\right) S_i(t)\mathrm{d}t + \sigma_i(t)S_i(t)\mathrm{d}B_i(t) \\
&= \left(R(t) + \sigma_i(t)\gamma_i(t)\right) S_i(t)\mathrm{d}t + \sigma_i(t)S_i(t)\mathrm{d}B_i(t) \\
&= R(t)S_i(t)\mathrm{d}t + \sigma_i(t)S_i(t)\left[\gamma_i(t)\mathrm{d}t + \mathrm{d}B_i(t)\right] \\
&= R(t)S_i(t)\mathrm{d}t + \sigma_i(t)S_i(t)\mathrm{d}\tilde{B}_i(t).
\end{aligned}$$

3. Notice that

$$\begin{aligned}
B_i(t) &= \sum_{j=1}^{d} \int_0^t \frac{\sigma_{ij}(u)}{\sigma_i(u)}\mathrm{d}W_j(u) \\
\tilde{B}_i(t) &= \sum_{j=1}^{d} \frac{\sigma_{ij}(t)}{\sigma_i(t)}\mathrm{d}\tilde{W}_j(t) \\
\mathrm{d}W_i(t)\mathrm{d}W_k(t) &= \mathrm{d}\tilde{W}_i(t)\mathrm{d}\tilde{W}_k(t)
\end{aligned}$$

Therefore, $\mathrm{d}B_i(t)\mathrm{d}B_k(t) = \mathrm{d}\tilde{B}_i(t)\mathrm{d}\tilde{B}_k(t)$.

4. We have that

$$\mathrm{d}B_i(t)B_k(t) = B_i(t)\mathrm{d}B_k(t) + B_k(t)\mathrm{d}B_i(t) + \rho_{ik}(t)\mathrm{d}t.$$

Thus,

$$B_i(t)B_k(t) = \int_0^t B_i(s)\mathrm{d}B_k(s) + \int_0^t B_k(s)\mathrm{d}B_i(s) + \int_0^t \rho_{ik}(s)\mathrm{d}s.$$

Taking expectation yields

$$\begin{aligned}\mathbb{E}[B_i(t)B_k(t)] &= \underbrace{\mathbb{E}\left[\int_0^t B_i(s)\mathrm{d}B_k(s)\right]}_{=0} + \underbrace{\mathbb{E}\left[\int_0^t B_k(s)\mathrm{d}B_i(s)\right]}_{=0} + \mathbb{E}\left[\int_0^t \rho_{ik}(s)\mathrm{d}s\right] \\ &= \mathbb{E}\left[\int_0^t \rho_{ik}(s)\mathrm{d}s\right] \\ &= \int_0^t \rho_{ik}(s)\mathrm{d}s\end{aligned}$$

Similarly,

$$\tilde{\mathbb{E}}[\tilde{B}_i(t)\tilde{B}_k(t)] = \int_0^t \rho_{ik}(s)\mathrm{d}s.$$

5. Consider the following

$$\begin{aligned}B_1(t) &= W_2(t) \\ B_2(t) &= \int_0^t \operatorname{sgn}(W_1(u))\mathrm{d}W_2(u) \\ \tilde{B}_1(t) &= B_1(t) \\ \tilde{B}_2(t) &= B_2(t)\end{aligned}$$

We want to show that

$$\mathbb{E}\left[\int_0^t \operatorname{sgn}(W_1(s))\mathrm{d}s\right] \neq \tilde{\mathbb{E}}\left[\int_0^t \operatorname{sgn}(W_1(s))\mathrm{d}s\right].$$

First, since $W_1(t)$ is a Brownian motion under $\mathbb{P}$,

$$\begin{aligned}\mathbb{E}\left[\int_0^t \operatorname{sgn}(W_1(s))\mathrm{d}s\right] &= \int_0^t \mathbb{E}\left[\operatorname{sgn}(W_1(s))\right]\mathrm{d}s \\ &= 0.\end{aligned}$$

Since $\tilde{W}_1(t) = W_1(t) + t$ and $\tilde{W}_1(t)$ is a Brownian motion under $\tilde{\mathbb{P}}$, it holds that

$$\begin{aligned}\tilde{\mathbb{E}}\left[\operatorname{sgn} W_1(t)\right] &= \tilde{\mathbb{E}}\left[\operatorname{sgn}\left(\tilde{W}_1(t) - t\right)\right] \\ &= \tilde{\mathbb{P}}\left(\tilde{W}_1(t) > t\right) - \tilde{\mathbb{P}}\left(\tilde{W}_1(t) < t\right) \\ &= 1 - 2\tilde{\mathbb{P}}\left(\tilde{W}_1(t) < t\right) \\ &= 1 - 2N(\sqrt{t}) \\ &< 0.\end{aligned}$$

Here $\tilde{W}_1(t) \sim \mathcal{N}(0,t)$ and so $\tilde{W}_1(t) = \sqrt{t}z$ with $z \sim \mathcal{N}(0,1)$. Therefore,

$$\begin{aligned}\tilde{\mathbb{E}}\left[\int_0^t \operatorname{sgn}(W_1(s))\mathrm{d}s\right] &= \int_0^t \tilde{\mathbb{E}}\left[\operatorname{sgn}(W_1(s))\right]\mathrm{d}s \\ &< 0.\end{aligned}$$

The desired conclusion therefore holds.