Exercise 5.13

Consider

$$\tilde{W}_1(t) = W_1(t)$$
$$\tilde{W}_2(t) = W_2(t) + \int_0^t W_1(s)\mathrm{d}s$$

Show that

$$\widetilde{\mathrm{Cov}}\left[W_1(T), W_2(T)\right] = -\frac{1}{2}T^2$$
$$\mathrm{Cov}[W_1(T), W_2(T)] = 0$$

Proof

We begin by noting that

$$\begin{aligned}
\widetilde{\mathrm{Cov}}\left[W_1(T), W_2(T)\right] &= \widetilde{\mathbb{E}}\left[W_1(T)W_2(T)\right] - \widetilde{\mathbb{E}}\left[W_1(T)\right]\widetilde{\mathbb{E}}\left[W_2(T)\right] \\
&= \widetilde{\mathbb{E}}\left[W_1(T)W_2(T)\right] - \underbrace{\widetilde{\mathbb{E}}\left[\tilde{W}_1(T)\right]}_{=0}\widetilde{\mathbb{E}}\left[\tilde{W}_2(T) - \int_0^T \tilde{W}_1(s)\mathrm{d}s\right] \\
&= \widetilde{\mathbb{E}}\left[\widetilde{W}_1(T)\left(\widetilde{W}_2(T) - \int_0^T \widetilde{W}_1(s)\mathrm{d}s\right)\right] \\
&= \widetilde{\mathbb{E}}\left[\widetilde{W}_1(T)\widetilde{W}_2(T)\right] - \widetilde{\mathbb{E}}\left[\widetilde{W}_1(T)\int_0^T \widetilde{W}_1(s)\mathrm{d}s\right] \\
&= \underbrace{\widetilde{\mathbb{E}}\left[\widetilde{W}_1(T)\right]}_{=0}\widetilde{\mathbb{E}}\left[\widetilde{W}_2(T)\right] - \widetilde{\mathbb{E}}\left[\widetilde{W}_1(T)\int_0^T \widetilde{W}_1(s)\mathrm{d}s\right] \\
&= -\widetilde{\mathbb{E}}\left[\widetilde{W}_1(T)\int_0^T \widetilde{W}_1(s)\mathrm{d}s\right] \\
&= -\int_0^T \widetilde{\mathbb{E}}\left[\widetilde{W}_1(T)\widetilde{W}_1(s)\right]\mathrm{d}s \\
&= -\int_0^T (T-s)\mathrm{d}s \\
&= -T^2 + \int_0^T s\mathrm{d}s \\
&= -\frac{T^2}{2}.
\end{aligned}$$

We did use the fact that $\widetilde{W}_1(t)$ and $\widetilde{W}_2(t)$ are independent under $\widetilde{\mathbb{P}}$. The second identity clearly holds as $W_1(T)$ and $W_2(T)$ are independent under $\mathbb{P}$.