Exercise 5.14 (Cost of carry)

Consider a commodity with value $S(t)$ at time. Cost of storage is equal to a per unit of commodity. Portfolio's differential is computed as below.

$$\mathrm{d}X(t) = \Delta(t)\mathrm{d}S(t) - a\Delta(t)\mathrm{d}t + r(X(t) - \Delta(t)S(t))\mathrm{d}t$$

The risk neutral measure is chosen so that

$$\mathrm{d}S(t) = rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}\tilde{W}(t) + a\mathrm{d}t$$

Discuss how to hedge a short position in a forward contract in this commodity.

Proof

We begin by showing that $e^{-rt}X(t)$ is a martingale. To this end, it holds that

$$\begin{aligned}
\mathrm{d}e^{-rt}X(t) &= e^{-rt}\left(-rX(t)\mathrm{d}t + \mathrm{d}X(t)\right) \\
&= e^{-rt}\Delta(t)\left(\mathrm{d}S(t) - a\mathrm{d}t - rS(t)\mathrm{d}t\right) \\
&= e^{-rt}\sigma\Delta(t)S(t)\mathrm{d}\tilde{W}(t)
\end{aligned}$$

Next, we will find a closed form formula for $S(t)$ in terms of Brownian motion $\tilde{W}(t)$. Denote

$$Y(t) = e^{\sigma\tilde{W}(t) + \left(r - \frac{1}{2}\sigma^2\right)t}$$

We will show that

$$S(t) = S(0)Y(t) + Y(t)\int_0^t \frac{a}{Y(s)}\mathrm{d}s.$$

To this end, first

$$\begin{aligned}
\mathrm{d}Y(t) &= Y(t)\cdot\left[\sigma\mathrm{d}\tilde{W}(t) + \left(r - \tfrac{1}{2}\sigma^2\right)\mathrm{d}t + \frac{1}{2\sigma^2\mathrm{d}t}\right] \\
&= Y(t)\cdot\left[r\mathrm{d}t + \sigma\mathrm{d}\tilde{W}(t)\right]
\end{aligned}$$

Next,

$$\begin{aligned}
\mathrm{d}Y(t)\left[S(0) + \int_0^t \frac{a}{Y(s)}\mathrm{d}s\right] &= Y(t)\cdot\frac{a}{Y(t)}\mathrm{d}t + \left[S(0) + \int_0^t \frac{a}{Y(s)}\mathrm{d}s\right]\mathrm{d}Y(t) \\
&= a\mathrm{d}t + \left[S(0) + \int_0^t \frac{a}{Y(s)}\mathrm{d}s\right]Y(t)\cdot\left[r\mathrm{d}t + \sigma\mathrm{d}\tilde{W}(t)\right] \\
&= a\mathrm{d}t + S(t)\cdot\left[r\mathrm{d}t + \sigma\mathrm{d}\tilde{W}(t)\right] \\
&= rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}\tilde{W}(t) + a\mathrm{d}t \\
&= \mathrm{d}S(t).
\end{aligned}$$

Thus, the desired equation holds. Next, we compute the future price for this commodity. We have

$$\tilde{\mathbb{E}}\left[S(T)|\mathcal{F}(t)\right] = S(0)\tilde{\mathbb{E}}\left[Y(T)|\mathcal{F}(t)\right] + \tilde{\mathbb{E}}\left[Y(T)|\mathcal{F}(t)\right]\int_0^t \frac{a}{Y(s)}\mathrm{d}s + a\int_t^T \tilde{\mathbb{E}}\left[\frac{Y(T)}{Y(s)}|\mathcal{F}(t)\right]\mathrm{d}s$$

Note

$$\begin{aligned}\tilde{\mathbb{E}}\left[Y(T)|\mathcal{F}(t)\right] &= e^{rT}\tilde{\mathbb{E}}\left[e^{-rT}Y(T)|\mathcal{F}(t)\right] \\ &= e^{rT}e^{-rt}Y(t) \\ &= e^{r(T-t)}Y(t)\end{aligned}$$

For $s \geq t$

$$\begin{aligned}\tilde{\mathbb{E}}\left[\frac{Y(T)}{Y(s)}|\mathcal{F}(t)\right] &= \tilde{\mathbb{E}}\left[\tilde{\mathbb{E}}\left[\frac{Y(T)}{Y(s)}|\mathcal{F}(s)\right]|\mathcal{F}(t)\right] \\ &= \tilde{\mathbb{E}}\left[\frac{1}{Y(s)}\tilde{\mathbb{E}}\left[Y(T)|\mathcal{F}(s)\right]|\mathcal{F}(t)\right] \\ &= \tilde{\mathbb{E}}\left[\frac{1}{Y(s)}e^{r(T-s)}Y(s)|\mathcal{F}(t)\right] \\ &= e^{r(T-s)}\end{aligned}$$

Putting pieces together,

$$\begin{aligned}\tilde{\mathbb{E}}\left[S(T)|\mathcal{F}(t)\right] &= e^{r(T-t)}Y(t)\underbrace{\left(a+\int_0^t \frac{a}{Y(s)}\mathrm{d}s\right)}_{=\frac{S(t)}{Y(t)}} + a\int_0^T e^{r(T-s)}\mathrm{d}s + a\int_t^T e^{r(T-s)}\mathrm{d}s \\ &= e^{r(T-t)}S(t) + a\int_t^T e^{r(T-s)}\mathrm{d}s\end{aligned}$$

Since $\int_t^T e^{r(T-s)}\mathrm{d}s = \int_0^T e^{r(T-s)}\mathrm{d}s - \int_0^t e^{r(T-s)}\mathrm{d}s$, it holds

$$\begin{aligned}\mathrm{d}\tilde{\mathbb{E}}\left[S(T)|\mathcal{F}(t)\right] &= \mathrm{d}e^{r(T-t)}S(t) - ae^{r(T-t)}\mathrm{d}t \\ &= e^{r(T-t)}\mathrm{d}S(t) - re^{r(T-t)}S(t) - ae^{r(T-t)}\mathrm{d}t \\ &= e^{r(T-t)}\left[\mathrm{d}S(t) - rS(t) - a\mathrm{d}t\right] \\ &= \sigma e^{r(T-t)}S(t)\mathrm{d}\tilde{W}(t)\end{aligned}$$

As expected, $\tilde{\mathbb{E}}\left[S(T)|\mathcal{F}(t)\right]$ is a martingale. Since the interest rate is constant, forward and future prices are equal. In fact,

$$\tilde{\mathbb{E}}\left[e^{-r(T-t)}\left(S(T)-K\right)|\mathcal{F}(t)\right] = 0 \iff K = \tilde{\mathbb{E}}\left[S(T)|\mathcal{F}(t)\right].$$

After shorting a forward contract, the agent buys one unit of the commodity, pays the cost of carry till maturity and delivers at time T which costs him $S(T)$ and benefits him $\mathrm{For}_S(0,T)$. To finance all of this, he invest or borrow in the money market account. The main purpose is to ensure that $X(T) = S(T) - \mathrm{For}_S(0,T)$ while $X(0) = 0$. This will happen since

$$\begin{aligned}\mathrm{d}X(t) &= \mathrm{d}S(t) - a\mathrm{d}t + r\left(X(t) - S(t)\right)\mathrm{d}t \\ &= \mathrm{d}S(t) - rS(t)\mathrm{d}t - a\mathrm{d}t + rX(t)\mathrm{d}t\end{aligned}$$

Thus,

$$\mathrm{d}e^{r(T-t)}X(t) = \mathrm{d}e^{r(T-t)}S(t) - ae^{r(T-t)}\mathrm{d}t$$

Finally,

$$\begin{aligned} X(T) = X(T) - e^{rT}X(0) &= S(T) - e^{rT}S(0) - a\int_0^T e^{r(T-s)}\mathrm{d}s \\ &= S(T) - \underset{S}{\mathrm{For}}(0,T). \end{aligned}$$