

Exercise 5.2

Recall the risk-neutral pricing formula:

$$D(t)V(t) = \tilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}(t)], \quad 0 \leq t \leq T.$$

Prove that it could be rewritten as below:

$$D(t)Z(t)V(t) = \mathbb{E}[D(T)Z(T)V(T)|\mathcal{F}(t)]$$

$D(t)Z(t)$ is called the *state price density process*.

Proof

We have that

$$\begin{aligned} D(t)Z(t)V(t) &= Z(t)\tilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}(t)] \\ &= \mathbb{E}[D(T)V(T)Z(T)|\mathcal{F}(t)] \end{aligned}$$

Here we used Lemma 5.2.2 with $(s, t) \mapsto (t, T)$ and $Y = D(T)V(T)$. Note that Y is indeed $\mathcal{F}(T)$ measurable.