Exercise 5.3

Remember that

$$\begin{aligned} c(0,x) &= \tilde{\mathbb{E}}\left[e^{-rT}\left(S(T)-K\right)^+\right] \\ &= \tilde{\mathbb{E}}\left[e^{-rT}\left(xe^{\sigma\tilde{W}(T)+\left(r-\frac{\sigma^2}{2}\right)T}-K\right)^+\right] \end{aligned}$$

(1) Let $h(s) = (s-K)^+$. Use formula for $h'(s)$ to obtain a formula for $c_x(0,x)$.

(2) Rewrite this formula in form of

$$c_x(0,x) = \hat{\mathbb{P}}(S(T) > K),$$

where $\hat{\mathbb{P}}$ is a probability measure equivalent to $\tilde{\mathbb{P}}$. Show that $\hat{W}(t) = \tilde{W}(t) - \sigma t$ is a Brownian motion under $\hat{\mathbb{P}}$.

(3) Conclude

$$\hat{\mathbb{P}}(S(T) > K) = \hat{\mathbb{P}}\left(-\frac{\hat{W}(T)}{\sqrt{T}} < d_+(T,x)\right) = N(d_+(T,x)).$$

Proof

(1) We have that

$$c(0,x) = \tilde{\mathbb{E}}[\underbrace{e^{-rT}h(S(T))}_{:=g(x)}]$$

Derivation w.r.t. x and noting that $g(x) \geq 0$, we obtain that

$$c(0,x) = \tilde{\mathbb{E}}[g(x)] \Rightarrow c_x(0,x) = \tilde{\mathbb{E}}\left[\frac{\partial g(x)}{\partial x}\right]$$

Also,

$$\frac{\partial g(x)}{\partial x} = e^{\sigma\tilde{W}(T)-\frac{\sigma^2 T}{2}} \cdot \mathbf{1}_{\{S(T)>K\}}$$

(2) Define $Z_\sigma(\omega) = \exp\left(\sigma\tilde{W}(T) - \frac{\sigma^2 T}{2}\right)$. Consider the probability measure given by

$$\mathrm{d}\hat{\mathbb{P}}(\omega) = Z_\sigma(\omega)\mathrm{d}\tilde{\mathbb{P}}(\omega).$$

Note that $\hat{W}(t) = \tilde{W}(t) - \sigma t$ is a Brownian motion under $\hat{\mathbb{P}}$. We have that

$$\hat{\mathbb{P}}(S(T) > K) = \tilde{\mathbb{E}}\left[\mathbf{1}_{\{S(T)>K\}} \cdot Z_\sigma(\omega)\right] = \tilde{\mathbb{E}}\left[\frac{\partial g(x)}{\partial x}\right] = c_x(0,x).$$

(3) Recall that

$$d_+(T,x) = \frac{1}{\sigma\sqrt{T}}\left[\log\frac{x}{K} + \left(r + \tfrac{1}{2}\sigma^2\right)T\right).$$

Therefore,

$$\begin{aligned}
S(T) > K &\iff xe^{\sigma \tilde{W}(T) + \left(r - \frac{\sigma^2}{2}\right)T} > K \\
&\iff \log \frac{x}{K} + \sigma \tilde{W}(T) + \left(r - \frac{\sigma^2}{2}\right) T > 0 \\
&\iff \log \frac{x}{K} + \sigma \hat{W}(T) + \left(r + \frac{\sigma^2}{2}\right) T > 0 \\
&\iff \sigma \hat{W}(T) + \sigma \sqrt{T} d_+(T, x) > 0 \\
&\iff d_+(T, x) > -\frac{\hat{W}(T)}{\sqrt{T}}
\end{aligned}$$

Since $\hat{W}(t)$ is a Brownian motion under $\hat{\mathbb{P}}$, $-\frac{\hat{W}(T)}{\sqrt{T}}$ is a standard normal distribution under $\hat{\mathbb{P}}$.