Exercise 5.4 (BSM formula for non-random interest rate and volatility)

Consider a stock price $S(t)$ satisfying

$$\mathrm{d}S(t) = r(t)S(t)\mathrm{d}t + \sigma(t)S(t)\mathrm{d}\tilde{W}(t).$$

Here $r(t)$ and $\sigma(t)$ are non-random. Show that

$$c(0, S(0)) = \mathrm{BSM}\left(T, S(0), K, \tfrac{1}{T}\int_0^T r(t)\mathrm{d}t, \sqrt{\tfrac{1}{T}\int_0^T \sigma^2(t)\mathrm{d}t}\right)$$

Proof

Recall that

$$\mathrm{BSM}\left(T, x, K, r, \sigma\right) := xN\left(\underbrace{\tfrac{1}{\sigma\sqrt{T}} \cdot \left[\log\tfrac{x}{K} + (r + \tfrac{\sigma^2}{2})T\right]}_{:=d_+}\right) - e^{-rT}KN\left(\underbrace{\tfrac{1}{\sigma\sqrt{T}}\left[\log\tfrac{x}{K} + (r - \tfrac{\sigma^2}{2})T\right]}_{:=d_-}\right)$$

And

$$c(0, S(0)) = \tilde{\mathbb{E}}\left[\exp\left(-\int_0^T r(t)\mathrm{d}t\right) \cdot (S(T) - K)^+\right]$$

We have that

$$S(T) = S(0)\exp\left(\underbrace{\int_0^T \sigma(t)\mathrm{d}\tilde{W}(t)}_{:=z} + \int_0^T \left(r(t) - \tfrac{1}{2}\sigma^2(t)\right)\mathrm{d}t\right)$$

z is an Itô integral with deterministic integrand. We know that such integrals are normal random variables and in our case

$$\mathrm{Var}(z) = \int_0^T \sigma^2(t)\mathrm{d}t \text{ and } \mu(z) = 0.$$

We need the following lemma.

Lemma 1. *Let $z_0 \sim N(\mu_0, \sigma_0^2)$ and $K > 0$. We have that*

$$\mathbb{E}(e^{z_0} - K)^+ = \exp\left(\mu_0 + \frac{\sigma_0^2}{2}\right) \cdot N\left(\sigma_0 - \frac{\log K - \mu_0}{\sigma_0}\right) - K \cdot N\left(\frac{\mu_0 - \log K}{\sigma_0}\right)$$

Proof of Lemma We have that

$$\begin{aligned}
\mathbb{E}(e^{z_0}-K)^+ &= \frac{1}{\sqrt{2\pi\sigma_0^2}}\cdot\int_{\log K}(e^t-K)\cdot\exp\left(-\frac{(t-\mu_0)^2}{2\sigma_0^2}\right)\mathrm{d}t\\
&= \frac{1}{\sqrt{2\pi\sigma_0^2}}\cdot\int_{\log K}(e^t-K)\cdot\exp\left(-\frac{(t-\mu_0)^2}{2\sigma_0^2}\right)\mathrm{d}t\\
&= \frac{1}{\sqrt{2\pi}}\cdot\int_{\frac{\log K-\mu_0}{\sigma_0}}(e^{(t+\mu_0)\sigma_0}-K)\cdot\exp\left(-\frac{t^2}{2}\right)\mathrm{d}t\\
&= \frac{1}{\sqrt{2\pi}}\cdot\int_{\frac{\log K-\mu_0}{\sigma_0}}e^{(t+\mu_0)\sigma_0}\cdot\exp\left(-\frac{t^2}{2}\right)\mathrm{d}t-K\cdot N\left(\frac{\mu_0-\log K}{\sigma_0}\right)\\
&= \frac{\exp\left(\mu_0+\frac{\sigma_0^2}{2}\right)}{\sqrt{2\pi}}\int_{\frac{\log K-\mu_0}{\sigma_0}}\exp\left(-\frac{(t-\sigma_0)^2}{2}\right)\mathrm{d}t-K\cdot N\left(\frac{\mu_0-\log K}{\sigma_0}\right)\\
&= \frac{\exp\left(\mu_0+\frac{\sigma_0^2}{2}\right)}{\sqrt{2\pi}}\int_{\frac{\log K-\mu_0}{\sigma_0}-\sigma_0}\exp\left(-\frac{t^2}{2}\right)\mathrm{d}t-K\cdot N\left(\frac{\mu_0-\log K}{\sigma_0}\right)\\
&= \exp\left(\mu_0+\frac{\sigma_0^2}{2}\right)\cdot N\left(\underbrace{\sigma_0-\frac{\log K-\mu_0}{\sigma_0}}_{:=d_+}\right)-K\cdot N\left(\underbrace{\frac{\mu_0-\log K}{\sigma_0}}_{:=d_-}\right)
\end{aligned}$$

Note that

$$\exp\left(\underbrace{\int_0^T r(t)\mathrm{d}t}_{:=rT}\right)\cdot c(0,S(0))=\underbrace{S(0)}_{:=x}\mathbb{E}_{z_0\sim N(\mu_0,\sigma_0^2)}\left(e^{z_0}-\frac{K}{S(0)}\right)^+$$

In view of our lemma, $\mu_0=\int_0^T\left(r(t)-\frac{1}{2}\sigma^2(t)\right)\mathrm{d}t$ and $\sigma_0^2=\int_0^T\sigma^2(t)\mathrm{d}t$. Define $\sigma^2:=\frac{1}{T}\int_0^T\sigma^2(t)\mathrm{d}t$. We obtain that

$$d_+=\frac{1}{\sigma\sqrt{T}}\left[\log\tfrac{x}{K}+rT-\tfrac{T}{2}\sigma^2+T\sigma^2\right],d_-=\frac{1}{\sigma\sqrt{T}}\left[rT-\tfrac{T}{2}\sigma^2+\log\tfrac{x}{K}\right]$$

The proof is complete.