Exercise 5.5

1. Denote
$$Z(t) = \exp\left(-\int_0^t \Theta(u)dW(u) - \tfrac{1}{2}\cdot\int_0^t \Theta^2(u)du\right)$$
Compute differential of $\frac{1}{Z(t)}$.

2. Let $\tilde{M}(t)$ be a martingale under $\tilde{\mathbb{P}}$. Show that $M(t) = Z(t)\tilde{M}(t)$ is a martingale under $\mathbb{P}$.

3. Write
$$M(t) = M(0) + \int_0^t \Gamma(u)dW(u), \quad \forall 0 \le t \le T$$
Write $\tilde{M}(t) = M(t)\cdot\frac{1}{Z(t)}$ and compute its differential using Itô product rule.

4. Show that for some adapted process $\tilde{\Gamma}(t)$, the following is true.
$$\tilde{M}(t) = \tilde{M}(0) + \int_0^t \tilde{\Gamma}(u)\mathrm{d}\tilde{W}(u)$$

Proof

1. Notice that
$$\frac{1}{Z(t)} = \exp\left(\int_0^t \Theta(u)dW(u) + \tfrac{1}{2}\cdot\int_0^t \Theta^2(u)du\right)$$
Let $X(t) = \int_0^t \Theta(u)dW(u) + \frac{1}{2}\cdot\int_0^t \Theta^2(u)du$ and $f(x) = e^x$. We have that $\frac{1}{Z(t)} = f(X(t))$. Thus,
$$\begin{aligned} df(X(t)) &= f'(X(t))dX(t) + \tfrac{1}{2}\cdot f''(X(t))dX(t)dX(t) \\ &= f(X(t))\cdot\left[\Theta(t)dW(t) + \Theta^2(t)dt\right] \end{aligned}$$

2. Since $\tilde{M}(t)$ is $\mathcal{F}(t)$ measurable, Lemma 5.2.2 gives
$$\tilde{M}(s) = \tilde{\mathbb{E}}[\tilde{M}(t)|\mathcal{F}(s)] = \frac{1}{Z(s)}\cdot\mathbb{E}[Z(t)\tilde{M}(t)|\mathcal{F}(s)]$$

3. We have that
$$\begin{aligned} \mathrm{d}\tilde{M}(t) &= \frac{1}{Z(t)}\mathrm{d}M(t) + M(t)\mathrm{d}\frac{1}{Z(t)} + \mathrm{d}M(t)\mathrm{d}\frac{1}{Z(t)} \\ &= \Gamma(t)\mathrm{d}W(t)\cdot\frac{1}{Z(t)} + M(t)\cdot\frac{1}{Z(t)}\cdot\left[\Theta(t)\mathrm{d}W(t) + \Theta^2(t)\mathrm{d}t\right] \\ &+ \Gamma(t)\mathrm{d}W(t)\frac{1}{Z(t)}\cdot\left[\Theta(t)\mathrm{d}W(t) + \Theta^2(t)\mathrm{d}t\right] \\ &= \frac{1}{Z(t)}\cdot\left[\Gamma(t)\mathrm{d}W(t) + M(t)\Theta(t)\mathrm{d}W(t) + \Theta^2(t)M(t)\mathrm{d}t + \Gamma(t)\Theta(t)\mathrm{d}t\right] \\ &= \frac{1}{Z(t)}\cdot\left[(\Gamma(t) + M(t)\Theta(t))\left(\mathrm{d}\tilde{W}(t) - \Theta(t)\mathrm{d}t\right) + \Theta^2(t)M(t)\mathrm{d}t + \Gamma(t)\Theta(t)\mathrm{d}t\right] \\ &= \underbrace{\frac{\Gamma(t) + M(t)\Theta(t)}{Z(t)}}_{:=\tilde{\Gamma}(t)}\cdot\mathrm{d}\tilde{W}(t) \end{aligned}$$

4. Taking integrals from $\mathrm{d}\tilde{M}(t) = \tilde{\Gamma}(t)\mathrm{d}\tilde{W}(t)$ result follows.