Exercise 5.6

Use two-dimensional Levy theorem to prove two-dimensional Girsanov theorem.

Proof

Levy theorem Under the following assumptions

- $M_1(t), M_2(t)$ for $t \geq 0$ are martingales relative to filtration $\mathcal{F}(t)$.
- $M_i(0) = 0$ and $M_i(t)$ has continuous paths.
- $\mathrm{d}M_i\mathrm{d}M_i = \mathrm{d}t$ and $\mathrm{d}M_1\mathrm{d}M_2 = 0$.

Levy theorem asserts that M_1 and M_2 are independent Brownian motions.

Girsanov theorem Consider the following assumptions and notations

- $W_1(t)$ and $W_2(t)$ are two Brownian motions under $\mathbb{P}$ and $\mathcal{F}(t)$ is its generated filtration
- $\Theta_1(t), \Theta_2(t)$ are two adapted processes
- Define

  $$Z(t) = \exp\left(-\int_0^t \Theta_1(u)\mathrm{d}W_1(u) + \Theta_2(u)\mathrm{d}W_2(u) - \tfrac{1}{2}\int_0^t \left(\Theta_1^2(t) + \Theta_2^2(t)\right)\mathrm{d}t\right)$$

  and suppose $\mathbb{E}\int_0^T \left(\Theta_1^2(t) + \Theta_2^2(t)\right) Z^2(t)\mathrm{d}t < +\infty$
- Set $\tilde{W}_i(t) = W_i(t) + \int_0^t \Theta_i(u)\mathrm{d}u$ and define $\tilde{\mathbb{P}}$ such that $\mathrm{d}\tilde{\mathbb{P}}(\omega) = Z(T)\mathrm{d}\mathbb{P}(\omega)$

  Girsanov theorem asserts that $\mathcal{W}(t) = \left(\tilde{W}_1(t), \tilde{W}_2(t)\right)$ is a two-dimensional Brownian motion under $\tilde{\mathbb{P}}$.

Proof

Denote $Z(t) = e^{X(t)}$. Therefore,

$$\mathrm{d}X(t) = -\Theta_1(t)\mathrm{d}W_1(t) - \Theta_2(t)\mathrm{d}W_2(t) - \tfrac{1}{2}\Theta_1^2(t)\mathrm{d}t - \tfrac{1}{2}\Theta_2^2(t)\mathrm{d}t, \mathrm{d}X(t)\mathrm{d}X(t) = \Theta_1^2(t)\mathrm{d}t + \Theta_2^2(t)\mathrm{d}t$$

Itô formula yields that

$$\begin{aligned}\mathrm{d}Z(t) &= e^{X(t)} \cdot [\mathrm{d}X(t) + \tfrac{1}{2}\mathrm{d}X(t)\mathrm{d}X(t)] \\ &= -e^{X(t)} \cdot [\Theta_1(t)\mathrm{d}W_1(t) + \Theta_2(t)\mathrm{d}W_2(t)] \\ &= -Z(t) \cdot [\Theta_1(t)\mathrm{d}W_1(t) + \Theta_2(t)\mathrm{d}W_2(t)]\end{aligned}$$

Integration gives

$$Z(t) = Z(0) - \int_0^t Z(u) \cdot [\Theta_1(u)\mathrm{d}W_1(u) + \Theta_2(u)\mathrm{d}W_2(u)].$$

Next, we show that $\tilde{W}_i(t)Z(t)$ is a martingale under $\mathbb{P}$. Note that

$$\begin{aligned} \mathrm{d}\tilde{W}_i(t)Z(t) &= \underbrace{\tilde{W}_i(t)\mathrm{d}Z(t)}_{\mathrm{d}t\text{-free}} + Z(t)\mathrm{d}\tilde{W}_i(t) + \mathrm{d}\tilde{W}_i(t)\mathrm{d}Z(t) \\ &= \underbrace{\tilde{W}_i(t)\mathrm{d}Z(t)}_{\mathrm{d}t\text{-free}} + \underbrace{Z(t)\mathrm{d}W_i(t)}_{\mathrm{d}t\text{-free}} + Z(t)\Theta_i(t)\mathrm{d}t + \mathrm{d}\tilde{W}_i(t)\mathrm{d}Z(t) \end{aligned}$$

Moreover,

$$\begin{aligned} \mathrm{d}\tilde{W}_i(t)\mathrm{d}Z(t) &= -\left(\mathrm{d}W_i(t) + \Theta_i(t)\mathrm{d}t\right) \cdot Z(t)[\Theta_1(t)\mathrm{d}W_1(t) + \Theta_2(t)\mathrm{d}W_2(t)] \\ &= -\mathrm{d}W_i(t) \cdot Z(t)[\Theta_1(t)\mathrm{d}W_1(t) + \Theta_2(t)\mathrm{d}W_2(t)] \\ &= -Z(t)\Theta_i(t)\mathrm{d}t. \end{aligned}$$

Therefore, $\mathrm{d}\tilde{W}_i(t)Z(t)$ is $\mathrm{d}t$-free. In other words,

$$\tilde{W}_i(T)Z(T) = \tilde{W}_i(0)Z(0) + \int_0^T [\cdots]\mathrm{d}W_1(u) + \int_0^T [\cdots]\mathrm{d}W_2(u)$$

Thus $\tilde{W}_i(t)Z(t)$ is a martingale under $\mathbb{P}$. Now similar to the one-dimensional case, it follows that $\tilde{W}_i(t)$ is a martingale under $\tilde{\mathbb{P}}$. It also immediately follows that

$$\begin{aligned} \mathrm{d}\tilde{W}_i(t)\mathrm{d}\tilde{W}_j(t) &= (\mathrm{d}W_i(t) + \Theta_i(t)\mathrm{d}t) \cdot (\mathrm{d}W_j(t) + \Theta_j(t)\mathrm{d}t) \\ &= \mathrm{d}W_i(t)\mathrm{d}W_j(t) \\ &= \delta_{ij}\mathrm{d}t. \end{aligned}$$

From two dimensional Levy theorem, it follows that $\mathcal{W}$ is a two-dimensional Brownian motion under $\tilde{\mathbb{P}}$. The proof is complete.