Exercise 5.7

Proof

We begin by recalling that in the multidimensional market model an agent's portfolio value satisfies

$$\mathrm{d}\left(D(t)X(t)\right) = \sum_{i=1}^{m} \frac{\Delta_i(t)}{D(t)} \mathrm{d}\left(D(t)S_i(t)\right)$$

In other words,

$$D(t)X(t) = X(0) + \sum_{i=1}^{m} \int_0^t \frac{\Delta_i(s)}{D(s)} \mathrm{d}\left(D(s)S_i(s)\right)$$

Suppose that $\Delta_i^{(1)}(t)$ and $\Delta_i^{(2)}(t)$ corresponds to portfolio $X_1(t)$ and $X_2(t)$. Therefore,

$$\mathbb{P}\left(\sum_{i=1}^{m} \int_0^T \frac{\Delta_i^{(1)}(s)}{D(s)} \mathrm{d}\left(D(s)S_i(s)\right) \geq 0\right) = 1 \text{ and } \mathbb{P}\left(\sum_{i=1}^{m} \int_0^T \frac{\Delta_i^{(1)}(s)}{D(s)} \mathrm{d}\left(D(s)S_i(s)\right) > 0\right) > 0.$$

It suffices to let $\Delta_i^{(1)}(t) = \Delta_i^{(2)}(t)$. In that case,

$$D(T)X_2(T) = X_2(0) + D(T)X_1(T)$$

Therefore,

$$\mathbb{P}(X_1(T) \geq 0) = 1 \;\&\; \mathbb{P}(X_1(T) > 0) > 0 \iff \mathbb{P}(D(T)X_2(T) \geq X_2(0)) = 1 \;\&\; \mathbb{P}(D(T)X_2(T) > X_2(0)) > 0.$$