Exercise 5.8 (Strictly positive assets are generalized GBM)

Let $W(t), 0 \leq t \leq T$ be a Brownian motion ad let $\mathcal{F}(t)$ be the resulting filtration. Consider an adapted process $\Theta(t)$ and define risk-neutral $\tilde{\mathbb{P}}$ and Brownian motion $\tilde{W}(t)$ by way of Girsanov theorem. Any martingale $\tilde{M}(t)$ under $\tilde{\mathbb{P}}$ has the following representation for some adapted process $\tilde{\Gamma}(t)$:

$$\tilde{M}(t) = \tilde{M}(0) + \int_0^t \tilde{\Gamma}(u)\mathrm{d}\tilde{B}(u), \quad 0 \leq t \leq T.$$

Let $V(T)$ be positive a.s. and $\mathcal{F}(T)$-measurable. Risk-neutral pricing asserts that the price at time t of a security paying $V(T)$ at time T is calculated as below:

$$V(t) = \tilde{\mathbb{E}}\left[e^{-\int_t^T R(u)du} \cdot V(T)|\mathcal{F}(t)\right], \quad 0 \leq t \leq T.$$

(1) Show that there exists adapted process $\tilde{\Gamma}(t)$ s.t.

$$\mathrm{d}V(t) = R(t)V(t)\mathrm{d}t + \frac{\tilde{\Gamma}(t)}{D(t)}\mathrm{d}\tilde{W}(t),$$

where $D(t) = e^{-\int_0^t R(u)du}$ is the discount process.

(2) For each $t \in [0, T]$, show that $V(t)$ is positive a.s.

(3) Prove that there exists adapted process $\sigma(t)$ s.t.

$$dV(t) = R(t)V(t)dt + \sigma(t)V(t)d\tilde{W}(t).$$

Proof

1. Notice that $D(t)V(t)$ is a martingale under $\tilde{\mathbb{P}}$ since

$$D(t)V(t) = \tilde{\mathbb{E}}\left[D(T) \cdot V(T)|\mathcal{F}(t)\right].$$

Using statement's intro, there exists adapted process $\tilde{\Gamma}(t)$ s.t.

$$V(t) = \frac{V(0)}{D(t)} + \frac{1}{D(t)} \cdot \int_0^t \tilde{\Gamma}(u)\mathrm{d}\tilde{W}(u)$$

Noting that $\mathrm{d}\frac{1}{D(t)} = \frac{R(t)}{D(t)}\mathrm{d}t$, we continue

$$\begin{aligned}
\mathrm{d}V(t) &= \frac{V(0)R(t)}{D(t)}\mathrm{d}t + \frac{R(t)}{D(t)}\mathrm{d}t \cdot \int_0^t \tilde{\Gamma}(u)\mathrm{d}\tilde{B}(u) + \frac{\tilde{\Gamma}(t)\mathrm{d}\tilde{W}(t)}{D(t)} \\
&= \frac{V(0)R(t)}{D(t)}\mathrm{d}t + R(t)\mathrm{d}t \cdot \left(V(t) - \frac{V(0)}{D(t)}\right) + \frac{\tilde{\Gamma}(t)\mathrm{d}\tilde{W}(t)}{D(t)} \\
&= R(t)V(t)\mathrm{d}t + \frac{\tilde{\Gamma}(t)\mathrm{d}\tilde{W}(t)}{D(t)}.
\end{aligned}$$

2. By assumption, $V(T) > 0$ a.s. Since expectation of a positive random variable is always positive, we conclude that $V(t) > 0$ a.s. for all $0 \leq t \leq T$.

3. Let

$$\sigma(t) := \frac{\tilde{\Gamma}(t)}{V(t)D(t)}$$

Since $V(t) > 0$ a.s., $\sigma(t)$ is well-defined. For clarity, let $\sigma(t) = 1$ whenever $V(t) = 0$.