Exercise 5.9 (Implying the risk-neutral distribution)

Consider $S(t)$ to be the price of an underlying asset. With $S(0) = x$, we have that

$$c(0, T, x, K) = \tilde{\mathbb{E}}\left[e^{-rT}\left(S(T) - K\right)^{+}\right]$$

Let $\tilde{p}(0, T, x, y)$ be the risk-neutral density in the y variable of the distribution of $S(T)$ where $S(0) = x$. We thus have that

$$c(0, T, x, K) = e^{-rT}\int_{K}^{+\infty}(y - K)\tilde{p}(0, T, x, y)\mathrm{d}y$$

Show that

- $c_K(0, T, x, K) = -e^{-rT}\int_K^{+\infty}\tilde{p}(0, T, x, y)\mathrm{d}y = -e^{-rT}\tilde{\mathbb{P}}\left(S(T) > K\right)$
- $c_{KK}(0, T, x, K) = e^{-rT}\tilde{p}(0, T, x, K)$

Proof

Recall the following formula from elementary calculus.

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{g(x)} f(x, t)\mathrm{d}t = f(x, g(x)) \cdot g'(x) + \int_0^{g(x)} \frac{\mathrm{d}f(x, t)}{\mathrm{d}x}\mathrm{d}t.$$

Note that

$$\int_K^{+\infty}(y - K)\tilde{p}(0, T, x, y)\mathrm{d}y = \int_0^{+\infty}(y - K)\tilde{p}(0, T, x, y)\mathrm{d}y - \int_0^{K}(y - K)\tilde{p}(0, T, x, y)\mathrm{d}y$$

The first and second integral have derivative $-\int_0^{+\infty}\tilde{p}(0, T, x, y)\mathrm{d}y$ and 0 respectively. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}K}\int_K^{+\infty}(y - K)\tilde{p}(0, T, x, y)\mathrm{d}y = -\int_0^{+\infty}\tilde{p}(0, T, x, y)\mathrm{d}y$$

And,

$$\frac{\mathrm{d}}{\mathrm{d}K}\int_0^{K}(y - K)\tilde{p}(0, T, x, y)\mathrm{d}y = -\int_0^{K}\tilde{p}(0, T, x, y)\mathrm{d}y$$

Summing two pieces together, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}K}\int_K^{+\infty}(y - K)\tilde{p}(0, T, x, y)\mathrm{d}y = -\int_K^{+\infty}\tilde{p}(0, T, x, y)\mathrm{d}y$$

By definition, $\int_K^{+\infty}\tilde{p}(0, T, x, y)\mathrm{d}y = \tilde{\mathbb{P}}\left(S(T) > K\right)$. The first item follows. To see the second item, we need to show that

$$\frac{\mathrm{d}}{\mathrm{d}K}\int_K^{+\infty}\tilde{p}(0, T, x, y)\mathrm{d}y = -\tilde{p}(0, T, x, K),$$

which also follows from the calculus identity above. Indeed,

$$\int_K^{+\infty}\tilde{p}(0, T, x, y)\mathrm{d}y = \int_0^{+\infty}\tilde{p}(0, T, x, y)\mathrm{d}y - \int_0^{K}\tilde{p}(0, T, x, y)\mathrm{d}y$$

The first integral is constant and hence has derivative zero. The second integral's derivative is equal to $\tilde{p}(0, T, x, K)$. Second item follows.