Exercise 6.1

Consider the following SDE

$$\mathrm{d}X(u) = (a(u) + b(u)X(u))\,\mathrm{d}u + (\gamma(u) + \sigma(u)X(u))\,\mathrm{d}W(u)$$

Here $W(u)$ is a Brownian motion relative to a filtration $\mathcal{F}(u)$ and $a(u), b(u), \gamma(u), \sigma(u)$ are adapted processes w.r.t. $\mathcal{F}(u)$. Fix an initial condition $X(t) = x$. Define

$$Z(u) = \exp\left(\int_t^u \sigma(v)\mathrm{d}W(v) + \int_t^u \left(b(v) - \tfrac{1}{2}\sigma^2(v)\right)\mathrm{d}v\right)$$

$$Y(u) = x + \int_t^u \frac{a(v) - \sigma(v)\gamma(v)}{Z(v)}\mathrm{d}v + \int_t^u \frac{\gamma(v)}{Z(v)}\mathrm{d}W(v)$$

Show that $X(u) = Y(u)Z(u)$ solves the described SDE.

Proof

Denote $\ell(u) := \int_t^u \sigma(v)\mathrm{d}W(v) + \int_t^u \left(b(v) - \frac{1}{2}\sigma^2(v)\right)\mathrm{d}v$ and let $f(\ell) = e^\ell$. For $u \geq t$,

$$\begin{aligned}
\mathrm{d}Z(u) = \mathrm{d}f(\ell(u)) &= f'(\ell(u))\mathrm{d}\ell(u) + \tfrac{1}{2}f''(\ell(u))\mathrm{d}\ell(u)\mathrm{d}\ell(u) \\
&= Z(u)\cdot\left[\sigma(u)\mathrm{d}W(u) + \left(b(u) - \tfrac{1}{2}\sigma^2(u)\right)\mathrm{d}u + \tfrac{1}{2}\sigma^2(u)\mathrm{d}u\right] \\
&= Z(u)\cdot[\sigma(u)\mathrm{d}W(u) + b(u)\mathrm{d}u]
\end{aligned}$$

Next,

$$\mathrm{d}Y(u) = \frac{a(u) - \sigma(u)\gamma(u)}{Z(u)}\mathrm{d}u + \frac{\gamma(u)}{Z(u)}\mathrm{d}W(u)$$

Therefore, $\mathrm{d}Y(u)Z(u)$ equals to

$$\underbrace{Z(u)\mathrm{d}Y(u)}_{(a(u)-\sigma(u)\gamma(u))\mathrm{d}u+\gamma(u)\mathrm{d}W(u)} + \underbrace{Y(u)\mathrm{d}Z(u)}_{Y(u)Z(u)\cdot[\sigma(u)\mathrm{d}W(u)+b(u)\mathrm{d}u]} + \underbrace{\mathrm{d}Y(u)\mathrm{d}Z(u)}_{[(a(u)-\sigma(u)\gamma(u))\mathrm{d}u+\gamma(u)\mathrm{d}W(u)]\cdot[\sigma(u)\mathrm{d}W(u)+b(u)\mathrm{d}u]}$$

Simplifying gives

$$\begin{aligned}
\mathrm{d}Y(u)Z(u) &= (a(u) - \sigma(u)\gamma(u))\,\mathrm{d}u + \gamma(u)\mathrm{d}W(u) + Y(u)Z(u)\cdot[\sigma(u)\mathrm{d}W(u) + b(u)\mathrm{d}u] + \gamma(u)\sigma(u)\mathrm{d}u \\
&= (a(u) + b(u)Y(u)Z(u))\,\mathrm{d}u + (\gamma(u) + \sigma(u)Y(u)Z(u))\,\mathrm{d}W(u)
\end{aligned}$$

Clearly, $Z(t) = 1$ and $Y(t) = x$. Thus, $Y(t)Z(t) = x$ and initial condition is satisfied.