Exercise 6.10 (Implying the volatility surface)

Suppose that a stock is governed by the following SDE

$$\mathrm{d}S(u) = rS(u)\mathrm{d}t + \sigma(u, S(u))S(u)\mathrm{d}\tilde{W}(u)$$

Denote by $\tilde{p}(t, T, x, y)$ the transition density. In particular, time-zero price of a call expiring at time T when $S(0) = x$ is equal to

$$c(0, T, x, K) = e^{-rT} \int_K^{+\infty} (y - K)\tilde{p}(0, T, x, y)\mathrm{d}y.$$

Assume the following conditions regarding tail of $\tilde{p}(t, T, x, y)$

- $\lim_{y\to\infty}(y - K)ry\tilde{p}(0, T, x, y) = 0$
- $\lim_{y\to\infty}(y - K)\frac{\partial}{\partial y}[\sigma^2(T, y)y^2\tilde{p}(0, T, x, y)] = 0$
- $\lim_{y\to\infty} \sigma^2(T, y)y^2\tilde{p}(0, T, x, y) = 0$

Show that the following equation holds

$$c_T(0, T, x, K) = -rKc_K(0, T, x, K) + \frac{1}{2}\sigma^2(T, K)K^2 c_{KK}(0, T, x, K)$$

Proof

We begin by noting that

$$c_T(0, T, x, K) = -rc(0, T, x, K) + e^{-rT} \int_K^{+\infty} (y - K)\frac{\partial}{\partial T}\tilde{p}(0, T, x, y)\mathrm{d}y.$$

Using Kolmogorov forward equation, we have that

$$\begin{aligned} c_T(0, T, x, K) &= -rc(0, T, x, K) \\ &\quad + e^{-rT} \int_K^{+\infty} (y - K)\left[-\frac{\partial}{\partial y}\left(ry\tilde{p}(0, T, x, y)\right) + \tfrac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(T, y)y^2\tilde{p}(0, T, x, y)\right)\right]\mathrm{d}y. \end{aligned}$$

Integration by parts implies that

$$\int_K^{+\infty} (y - K)\frac{\partial}{\partial y}\left(ry\tilde{p}(0, T, x, y)\right)\mathrm{d}y + \int_K^{+\infty} ry\tilde{p}(0, T, x, y)\mathrm{d}y = (y - K)ry\tilde{p}(0, T, x, y)|_K^{+\infty}$$

By first regularity condition, right hand side is zero. Therefore,

$$-\int_K^{+\infty} (y - K)\frac{\partial}{\partial y}\left(ry\tilde{p}(0, T, x, y)\right)\mathrm{d}y = \int_K^{+\infty} ry\tilde{p}(0, T, x, y)\mathrm{d}y$$

Integration by parts and using the second regularity condition above gives

$$\begin{aligned} \int_K^{+\infty} (y - K)\frac{\partial^2}{\partial y^2}\left(\sigma^2(T, y)y^2\tilde{p}(0, T, x, y)\right)\mathrm{d}y + \int_K^{+\infty} \frac{\partial}{\partial y}\left(\sigma^2(T, y)y^2\tilde{p}(0, T, x, y)\right)\mathrm{d}y \\ = (y - K)\frac{\partial}{\partial y}\left(\sigma^2(T, y)y^2\tilde{p}(0, T, x, y)\right)|_K^{+\infty} \\ = 0. \end{aligned}$$

Therefore,

$$\begin{aligned}\int_K^{+\infty} (y-K)\frac{\partial^2}{\partial y^2}\left(\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)\right)\mathrm{d}y &= -\int_K^{+\infty}\frac{\partial}{\partial y}\left(\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)\right)\mathrm{d}y \\ &= \sigma^2(T,K)K^2\tilde{p}(0,T,x,K)\end{aligned}$$

Here we used the last regularity condition. We next have that

$$\begin{aligned}&-rc(0,T,x,K) - e^{-rT}\int_K^{+\infty}(y-K)\frac{\partial}{\partial y}\left(ry\tilde{p}(0,T,x,y)\right) \\ &= -re^{-rT}\int_K^{+\infty}(y-K)\tilde{p}(0,T,x,y)\mathrm{d}y + e^{-rT}\int_K^{+\infty} ry\tilde{p}(0,T,x,y)\mathrm{d}y \\ &= rKe^{-rT}\int_K^{+\infty}\tilde{p}(0,T,x,y)\mathrm{d}y \\ &= -rKc_K(0,T,x,K)\end{aligned}$$

Last equality is driven from Ex. 5.9. Putting pieces together, we obtain that

$$\begin{aligned}c_T(0,T,x,K) &= -rKc_K(0,T,x,K) + \tfrac{1}{2}e^{-rT}\sigma^2(T,K)K^2\tilde{p}(0,T,x,K) \\ &= -rKc_K(0,T,x,K) + \tfrac{1}{2}\sigma^2(T,K)K^2c_{KK}(0,T,x,K).\end{aligned}$$