### Exercise 6.2 (No-arbitrage derivation of bond-pricing equation)

Suppose that interest rate is provided by the following SDE

$$\mathrm{d}R(t) = \alpha(t, R(t))\mathrm{d}t + \gamma(t, R(t))\mathrm{d}W(t).$$

Bond-pricing equation driven via risk-neutral pricing is given as follows

$$f_t(t, r, T) + \beta(t, r)f_r(t, r, T) + \tfrac{1}{2}\gamma^2(t, r)f_{rr}(t, r, T) = rf(t, r, T).$$

Here $f(t, r, T) = \tilde{\mathbb{E}}[e^{-\int_t^T R(s)\mathrm{d}s}|\mathcal{F}(t)]$, $B(t, T) = f(t, R(t), T)$, and $\beta(t, R(t))$ denotes the drift under risk-neutral measure inside interest rate's SDE. Derive this equation using a no-arbitrage argument.

### Proof

Define $\beta(t, r, T)$ such that

$$f_t(t, r, T) + \beta(t, r, T)f_r(t, r, T) + \tfrac{1}{2}\gamma^2(t, r)f_{rr}(t, r, T) = rf(t, r, T).$$

This is readily possible if $f_r(t, r, T) \neq 0$. If $f_r(t, r, T) = 1$, then set $\beta(t, r, T_i) = 1$. Consider the following portfolio value process for $t \in [0, T_1]$: At time $t$ holds

- $\Delta_1(t)$ of bonds maturing at time $T_1$
- $\Delta_2(t)$ of bonds maturing at time $T_2$
- Borrow or invest in the money market account if necessary

Note that

$$\mathrm{d}X(t) = \sum_{i=1}^{2} \Delta_i(t)\mathrm{d}f(t, r, T_i) + R(t)\left(X(t) - \sum_{i=1}^{2} \Delta_i(t)f(t, r, T_i)\right)\mathrm{d}t.$$

Therefore,

$$\begin{aligned}
\mathrm{d}D(t)X(t) &= D(t)\mathrm{d}X(t) + X(t)\mathrm{d}D(t) \\
&= D(t)\mathrm{d}X(t) - X(t)D(t)R(t)\mathrm{d}t \\
&= D(t)\left[\sum_{i=1}^{2} \Delta_i(t)\mathrm{d}f(t, r, T_i) + R(t)\left(X(t) - \sum_{i=1}^{2} \Delta_i(t)f(t, r, T_i)\right)\mathrm{d}t - X(t)R(t)\mathrm{d}t\right] \\
&= D(t)\sum_{i=1}^{2} \Delta_i(t)\left[\mathrm{d}f(t, r, T_i) - R(t)f(t, r, T_i)\mathrm{d}t\right].
\end{aligned}$$

We have that

$$\begin{aligned}
\mathrm{d}f(t, r, T_i) &= f_t(t, r, T_i)\mathrm{d}t + f_r(t, r, T_i)\mathrm{d}R(t) + \frac{1}{2}f_{rr}(t, r, T_i)\mathrm{d}R(t)\mathrm{d}R(t) \\
&= f_t(t, r, T_i)\mathrm{d}t + \alpha(t, r)f_r(t, r, T_i)\mathrm{d}t + \gamma(t, r)f_r(t, r, T_i)\mathrm{d}W(t) + \frac{1}{2}f_{rr}(t, r, T_i)\mathrm{d}R(t)\mathrm{d}R(t) \\
&= f_t(t, r, T_i)\mathrm{d}t + \alpha(t, r)f_r(t, r, T_i)\mathrm{d}t + \gamma(t, r)f_r(t, r, T_i)\mathrm{d}W(t) + \frac{1}{2}\gamma^2(t, r)f_{rr}(t, r, T_i)\mathrm{d}t
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathrm{d}f(t,r,T_i) - R(t)f(t,r,T_i)\mathrm{d}t \\
&\qquad = f_t(t,r,T_i)\mathrm{d}t + \alpha(t,r)f_r(t,r,T_i)\mathrm{d}t + \gamma(t,r)f_r(t,r,T_i)\mathrm{d}W(t) + \frac{1}{2}\gamma^2(t,r,T_i)f_{rr}(t,r,T_i)\mathrm{d}t \\
&\qquad - f_t(t,r,T_i)\mathrm{d}t - \beta(t,r,T_i)f_r(t,r,T_i)\mathrm{d}t - \tfrac{1}{2}\gamma^2(t,r,T_i)f_{rr}(t,r,T_i)\mathrm{d}t \\
&\qquad = f_r(t,r,T_i)\left[\alpha(t,r) - \beta(t,r,T_i)\right]\mathrm{d}t + \gamma(t,r)f_r(t,r,T_i)\mathrm{d}W(t)
\end{aligned}$$

We now construct an arbitrage portfolio if $\beta(t,R(t),T_1) \neq \beta(t,R(t),T_2)$. We say a portfolio value process $X(t)$ satisfying $X(0) = 0$ is an arbitrage if for some time $T > 0$

$$\mathbb{P}(X(T) \geq 0) = 1 \text{ and } \mathbb{P}(X(T) > 0) > 0.$$

Define

$$M(t) = \left[\beta(t,R(t),T_2) - \beta(t,R(t),T_1)\right] f_r(t,R(t),T_1)f_r(t,R(t),T_2)$$

Set

$$S(t) = \operatorname{sign} M(t)$$

where sign $\in \{1,-1,0\}$. Denote

$$\Delta_i(t) = (-1)^{i+1}S(t)f_r(t,R(t),T_j) \text{ where } 1 \leq i \neq j \leq 2.$$

Note that

$$\begin{aligned}
\mathrm{d}\left(D(t)X(t)\right) &= D(t)\sum_{i=1}^{2}\Delta_i(t)\left[\mathrm{d}f(t,r,T_i) - R(t)f(t,r,T_i)\mathrm{d}t\right] \\
&= D(t)\sum_{i=1}^{2}\Delta_i(t)\left(f_r(t,r,T_i)\left[\alpha(t,r) - \beta(t,r,T_i)\right]\mathrm{d}t + \gamma(t,r)f_r(t,r,T_i)\mathrm{d}W(t)\right) \\
&= D(t)\sum_{i=1}^{2}(-1)^{i+1}S(t)f_r(t,R(t),T_j)\left(f_r(t,r,T_i)\left[\alpha(t,r) - \beta(t,r,T_i)\right]\mathrm{d}t + \gamma(t,r)f_r(t,r,T_i)\mathrm{d}W(t)\right) \\
&= D(t)S(t)f_r(t,R(t),T_1)f_r(t,R(t),T_2)\sum_{i=1}^{2}(-1)^{i+1}\left[\left[\alpha(t,r) - \beta(t,r,T_i)\right]\mathrm{d}t + \gamma(t,r)\mathrm{d}W(t)\right] \\
&= D(t)S(t)f_r(t,R(t),T_1)f_r(t,R(t),T_2)\left[\beta(t,r,T_2) - \beta(t,r,T_1)\right]\mathrm{d}t \\
&= D(t)\underbrace{|M(t)|}_{:=\mu(t)}\mathrm{d}t.
\end{aligned}$$

Thus,

$$\mathrm{d}\left(D(t)X(t)\right) = \mu(t)D(t)\mathrm{d}t \text{ for some } \mu(t) \geq 0.$$

Hence, to ensure $X$ does not result in an arbitrage opportunity, we must have that $\mu(t) = 0$ a.s. Therefore, $\beta(t,r,T)$ does not depend on $T$ and we have the same ODE for pricing bonds that was obtained via risk-neutral pricing. Finally, if we define

$$\tilde{W}(t) = W(t) + \int_0^t \frac{\alpha(u,R(u)) - \beta(u,R(u))}{\gamma(u,R(u))}\mathrm{d}u.$$

Thus,

$$\begin{aligned} \mathrm{d}f(t,r,T) - R(t)f(t,r,T)\mathrm{d}t &= f_r(t,r,T)\left[\alpha(t,r) - \beta(t,r,T)\right]\mathrm{d}t + \gamma(t,r)f_r(t,r,T)\mathrm{d}W(t) \\ &= \gamma(t,r)f_r(t,r,T)\mathrm{d}\tilde{W}(t) \end{aligned}$$

In other words, fixating $T$ and omitting it from the equation,

$$\begin{aligned} \mathrm{d}D(t)f(t,r) &= D(t)\mathrm{d}f(t,r) - f(t,r)D(t)R(t)\mathrm{d}t \\ &= D(t)\left(\mathrm{d}f(t,r) - f(t,r)R(t)\mathrm{d}t\right) \\ &= D(t)\gamma(t,r)f_r(t,r)\mathrm{d}\tilde{W}(t) \end{aligned}$$

Thus, $\tilde{\mathbb{P}}$ under which $\tilde{W}$ is a Brownian motion is risk neutral.