Exercise 6.3 (Solution of Hull-White model)

In Hull-White model, we assume the following SDE

$$\mathrm{d}R(t) = \beta(t, R(t))\mathrm{d}t + \gamma(t, R(t))\mathrm{d}\tilde{W}(t)$$

From this equation, we drive the PDE for the zero-coupon bond price

$$f_t(t,r) + (a(t) - b(t)r)\, f_r(t,r) + \tfrac{1}{2}\sigma^2(t) f_{rr}(t,r) = r f(t,r)$$

which has its solution of the following form

$$f(t,r) = e^{-rC(t,T)-A(t,T)} \text{ where } C(T,T) = A(T,T) = 0.$$

Derive and insert back to the above PDE, we obtain that

$$C'(t,T) = b(t)C(t,T) - 1 \text{ and } A'(t,T) = -a(t)C(t,T) + \tfrac{1}{2}\sigma^2(t)C^2(t,T).$$

Show that

$$C(t,T) = \int_t^T e^{-\int_t^s b(v)\mathrm{d}v}\mathrm{d}s \text{ and } A(t,T) = \int_t^T \left(a(s)C(s,T) - \tfrac{1}{2}\sigma^2(s)C^2(s,T)\right)\mathrm{d}s$$

Proof

Recall that

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{f(x)} g(t)\mathrm{d}t = f'(x)g(f(x))$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{s} b(v)\mathrm{d}v = b(s).$$

We have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\left[e^{-\int_0^s b(v)\mathrm{d}v}C(s,T)\right] &= -b(s)e^{-\int_0^s b(v)\mathrm{d}v}C(s,T) + e^{-\int_0^s b(v)\mathrm{d}v}C'(s,T) \\
&= e^{-\int_0^s b(v)\mathrm{d}v}\left[-b(s)C(s,T) + C'(s,T)\right] \\
&= -e^{-\int_0^s b(v)\mathrm{d}v}
\end{aligned}$$

Thus,

$$e^{-\int_0^T b(v)\mathrm{d}v}C(T,T) - e^{-\int_0^t b(v)\mathrm{d}v}C(t,T) = -\int_t^T e^{-\int_0^s b(v)\mathrm{d}v}\mathrm{d}s$$

Using $C(T,T) = 0$ and cancelling out $-e^{-\int_0^t b(v)\mathrm{d}v}$ from both sides, we obtain

$$C(t,T) = \int_t^T e^{-\int_t^s b(v)\mathrm{d}v}\mathrm{d}s.$$

Integration from t to T and considering $A(T,T) = 0$ immediately gives the solution for $A(t,T)$.