Exercise 6.4 (Solution of Cox-Ingersoll-Ross model)

Recall that

$$C'(t,T) = bC(t,T) + \tfrac{1}{2}\sigma^2 C^2(t,T) - 1, \quad A'(t,T) = -aC(t,T).$$

Terminal conditions are $C(T,T) = A(T,T) = 0$. Let $\gamma = \frac{1}{2}\sqrt{b^2 + 2\sigma^2}$ and $\tau = T - t$. Show that the following equations hold

$$C(t,T) = \frac{\sinh \gamma\tau}{\gamma \cosh \gamma\tau + \frac{1}{2}\sinh \gamma\tau}, \quad A(t,T) = -\tfrac{2a}{b^2} \log \frac{\gamma e^{\frac{1}{2}b\tau}}{\gamma \cosh \gamma\tau + \frac{1}{2}b \sinh \gamma\tau}.$$

Proof

Define

$$\varphi(t) = \exp\left(\tfrac{\sigma^2}{2}\int_t^T C(u,T)\mathrm{d}u\right)$$

Therefore,

$$\ln \varphi(t) = \tfrac{\sigma^2}{2}\int_t^T C(u,T)\mathrm{d}u \Rightarrow \frac{\varphi'(t)}{\varphi(t)} = -\tfrac{\sigma^2}{2}C(t,T)$$

From elementary calculus $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{g(x)} f(t)\mathrm{d}t = f(g(x))) \cdot g'(x)$ and $\int_t^T C(u,T)\mathrm{d}u = \int_T^t -C(u,T)\mathrm{d}u$. Continuing, we have that

$$\begin{aligned}
\varphi''(t) &= -\tfrac{\sigma^2}{2}C'(t,T)\cdot\varphi(t) - \tfrac{\sigma^2}{2}C(t,T)\left[-\tfrac{\sigma^2}{2}C(t,T)\cdot\varphi(t)\right] \\
&= -\tfrac{\sigma^2}{2}\varphi(t)\cdot\left[C'(t,T) - \tfrac{\sigma^2}{2}C^2(t,T)\right] \\
&= -\tfrac{\sigma^2}{2}\varphi(t)\cdot\left[bC(t,T) - 1\right].
\end{aligned}$$

Therefore,

$$\frac{\varphi''(t)}{\varphi(t)} - b\cdot\frac{\varphi'(t)}{\varphi(t)} - \frac{\sigma^2}{2} = -\frac{\sigma^2}{2}\cdot\left[bC(t,T) - 1 - bC(t,T) + 1\right]$$

Thus, $\varphi(t)$ must satisfy $\varphi(t) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}$ where λ_1, λ_2 are roots of the quadratic equation

$$\lambda^2 - b\lambda - \tfrac{\sigma^2}{2} = 0.$$

Since $\lambda_1 + \lambda_2 = b$, for $\gamma > 0$, we let

$$\lambda_1 = \tfrac{b}{2} - \gamma \text{ and } \lambda_2 = \tfrac{b}{2} + \gamma$$

Therefore, $-\frac{\sigma^2}{2} = \lambda_1\lambda_2 = \frac{b^2}{4} - \gamma^2$. Thus, $\gamma = \frac{1}{2}\sqrt{b^2 + 2\sigma^2}$. Next, denote

$$c_1 = -\lambda_1 a_1 e^{\lambda_1 T}, \quad c_2 = \lambda_2 a_2 e^{\lambda_2 T}.$$

Therefore,

$$1 = \varphi(T) = a_1 e^{\lambda_1 T} + a_2 e^{\lambda_2 T} = \frac{c_2}{\lambda_2} - \frac{c_1}{\lambda_1}$$

Next, since $\varphi'(T) = -\frac{\sigma^2}{2}\varphi(T)C(T,T) = 0$, we must have

$$0 = \varphi'(T) = \lambda_1 a_1 e^{\lambda_1 T} + \lambda_2 a_2 e^{\lambda_2 T} = c_2 - c_1.$$

Letting $c = c_1 = c_2$,

$$\varphi(t) = c \cdot \frac{e^{-\left(\frac{b}{2}+\gamma\right)\tau}}{\frac{b}{2}+\gamma} - c \cdot \frac{e^{-\left(\frac{b}{2}-\gamma\right)\tau}}{\frac{b}{2}-\gamma}$$

Derivation gives

$$\varphi'(t) = c \cdot e^{-\left(\frac{b}{2}+\gamma\right)\tau} - c \cdot e^{-\left(\frac{b}{2}-\gamma\right)\tau}.$$

Putting pieces together and using $C(t,T) = -\frac{2\varphi'(t)}{\sigma^2\varphi(t)}$, we derive the expression for $C(t,T)$. Similarly, since $A'(t,T) = -aC(t,T) = \frac{2a\varphi'(t)}{\sigma^2\varphi(t)}$, we obtain that

$$\underbrace{\frac{2a}{\sigma^2}\ln\varphi(T)}_{=0} - \frac{2a}{\sigma^2}\ln\varphi(t) = \underbrace{A(T,T)}_{=0} - A(t,T).$$

A simple manipulation gives the results.