Exercise 6.5 (Two-dimensional Feynman-Kac)

Consider

$$\mathrm{d}X_1(t) = \beta_1(t, X_1(t), X_2(t))\mathrm{d}t + \gamma_{11}(t, X_1(t), X_2(t))\mathrm{d}W_1(t) + \gamma_{12}(t, X_1(t), X_2(t))\mathrm{d}W_2(t)$$
$$\mathrm{d}X_2(t) = \beta_2(t, X_1(t), X_2(t))\mathrm{d}t + \gamma_{21}(t, X_1(t), X_2(t))\mathrm{d}W_1(t) + \gamma_{22}(t, X_1(t), X_2(t))\mathrm{d}W_2(t)$$

Set the initial conditions to be $X_1(t) = x_1$ and $X_2(t) = x_2$. Let $h(y_1, y_2)$ be a Borel measurable function. Define

$$g(t, x_1, x_2) = \mathbb{E}^{t,x_1,x_2} h(X_1(T), X_2(T))$$
$$f(t, x_1, x_2) = \mathbb{E}^{t,x_1,x_2} e^{-r(T-t)} h(X_1(T), X_2(T))$$

Assume that $\mathrm{d}W_1(t)\mathrm{d}W_2(t) = \rho \mathrm{d}t$. Prove that

$$\begin{aligned} g_t + \beta_1 g_{x_1} + \beta_2 g_{x_2} & \\ &+ \left(\tfrac{1}{2}\gamma_{11}^2 + \rho\gamma_{11}\gamma_{12} + \tfrac{1}{2}\gamma_{12}^2\right) g_{x_1x_1} \\ &+ \left(\gamma_{11}\gamma_{21} + \rho\gamma_{11}\gamma_{22} + \rho\gamma_{12}\gamma_{22} + \tfrac{1}{2}\gamma_{22}^2\right) g_{x_2x_2} = 0. \end{aligned}$$

Similar equation holds for f except RHS $= rf$ as in Discounted Feynman-Kac Theorem.

Proof

We begin by noting that $g(t, X_1(t), X_2(t))$ and $f(t, X_1(t), X_2(t))$ are martingales. For example, for $0 \le s < t$, using Theorem 6.3.1.

$$g(t, X_1(t), X_2(t)) = \mathbb{E}[h(X_1(T), X_2(T))|\mathcal{F}(t)]$$

Therefore,

$$\begin{aligned} \mathbb{E}[g(t, X_1(t), X_2(t))|\mathcal{F}(s)] &= \mathbb{E}[\mathbb{E}[h(X_1(T), X_2(T))|\mathcal{F}(t)]|\mathcal{F}(s)] \\ &= \mathbb{E}[h(X_1(T), X_2(T))|\mathcal{F}(s)] \\ &= g(s, X_1(s), X_2(s)). \end{aligned}$$

Same argument works for $e^{-rt}f$. Differentiation gives

$$\mathrm{d}g = g_t\mathrm{d}t + g_{x_1}\mathrm{d}X_1 + g_{x_2}\mathrm{d}X_2 + \tfrac{1}{2}g_{x_1x_1}\mathrm{d}X_1\mathrm{d}X_1 + \tfrac{1}{2}g_{x_2x_2}\mathrm{d}X_2\mathrm{d}X_2 + g_{x_1x_2}\mathrm{d}X_1\mathrm{d}X_2$$

Keeping $\mathrm{d}t$ terms only gives

$$\begin{aligned} \mathrm{d}t\text{-term inside } \mathrm{d}X_1 &= \beta_1\mathrm{d}t \\ \mathrm{d}t\text{-term inside } \mathrm{d}X_2 &= \beta_2\mathrm{d}t \\ \mathrm{d}t\text{-term inside } \mathrm{d}X_1\mathrm{d}X_1 &= \gamma_{11}^2 + \gamma_{12}^2 + \rho\gamma_{11}\gamma_{12} \\ \mathrm{d}t\text{-term inside } \mathrm{d}X_2\mathrm{d}X_2 &= \gamma_{21}^2 + \gamma_{22}^2 + \rho\gamma_{21}\gamma_{22} \\ \mathrm{d}t\text{-term inside } \mathrm{d}X_1\mathrm{d}X_2 &= \gamma_{11}\gamma_{21} + \gamma_{12}\gamma_{22} + \rho\gamma_{11}\gamma_{22} + \rho\gamma_{21}\gamma_{12} \end{aligned}$$

Equating the $\mathrm{d}t$-term inside $\mathrm{d}g$, the result immediately follows.