Exercise 6.6 (MGF for CIR process)

Consider $W_1, \cdots, W_d$ to be independent Brownian motions and let $b, \sigma > 0$. For $j \in [1, d]$, define the Ornstein Uhlenbeck SDE as below

$$\mathrm{d}X_j(t) = -\tfrac{b}{2}X_j(t)\mathrm{d}t + \tfrac{1}{2}\sigma \mathrm{d}W_j(t)$$

1. Show that

$$X_j(t) = e^{-\frac{bt}{2}}\left[X_j(0) + \frac{\sigma}{2}\int_0^t e^{\frac{bu}{2}}\mathrm{d}W_j(u)\right]$$

   Moreover, for fixed t, it must hold that

$$\mathbb{E}X_j(t) = e^{-\frac{bt}{2}}X_j(0), \quad \operatorname{Var} X_j(t) = \frac{\sigma^2}{4b}\left(1 - e^{-bt}\right)$$

2. Define $R(t) = \sum_{j=1}^d X_j^2(t)$ and let $B(t) = \sum_{j=1}^d \int_0^t \frac{X_j(s)}{\sqrt{R(s)}}\mathrm{d}W_j(s)$. Show that $B(t)$ is a Brownian motion and

$$\mathrm{d}R(t) = (a - bR(t))\,\mathrm{d}t + \sigma\sqrt{R(t)}\mathrm{d}B(t) \quad \text{where } a = \frac{d\sigma^2}{4}.$$

3. Assume that $R(0) > 0$ and let $X_j(0) = \sqrt{\frac{R(0)}{d}}$. Show that $X_j(t)$ for $j \in [1, d]$ are *i.i.d* normal random variables. Denote $\mu(t) = \mathbb{E}X_j(t)$ and $v(t) = \operatorname{Var} X_j(t)$.

4. Show that

$$\mathbb{E}\exp\left(uX_j^2(t)\right) = \frac{1}{\sqrt{1 - 2v(t)u}}\exp\left(\frac{u\mu^2(t)}{1 - 2v(t)u}\right)$$

   Note that

$$\mathbb{E}e^{uR(t)} = \Pi_{j=1}^d \mathbb{E}e^{uX_j^2(t)} = \left(\mathbb{E}e^{uX_1^2(t)}\right)^d.$$

Proof

1. Notice that

$$\begin{aligned}
\mathrm{d}e^{\frac{bt}{2}}X_j(t) &= \tfrac{b}{2}e^{\frac{bt}{2}}X_j(t)\mathrm{d}t + e^{\frac{bt}{2}}\mathrm{d}X_j(t) \\
&= e^{\frac{bt}{2}} \cdot \left(\tfrac{b}{2}X_j(t)\mathrm{d}t + \mathrm{d}X_j(t)\right) \\
&= \tfrac{\sigma}{2}e^{\frac{bt}{2}}\mathrm{d}W_j(t).
\end{aligned}$$

   Integration from both sides concludes the first part. For fixed t, $\int_0^t e^{\frac{bu}{2}}\mathrm{d}W_j(u)$ is an Itô integral with deterministic integrand. Moreover,

$$\mathbb{E}\int_0^t e^{\frac{bu}{2}}\mathrm{d}W_j(u) = 0 \text{ and } \operatorname{Var}\int_0^t e^{\frac{bu}{2}}\mathrm{d}W_j(u) = \int_0^t e^{bu}\mathrm{d}u = \frac{e^{bt}}{b}\cdot\left(1 - e^{-bt}\right)$$

   The second part immediately follows.

2. $B(t)$ is sum of Itô integrals and hence a martingale. Next,

$$\mathrm{d}B(t) = \sum_{j=1}^{d} \frac{X_j(t)}{\sqrt{R(t)}} \mathrm{d}W_j(t).$$

Since $\mathrm{d}W_j(t)\mathrm{d}W_i(t) = \delta_{ij}\mathrm{d}t$, we have that

$$\mathrm{d}B(t)\mathrm{d}B(t) = \sum_{j=1}^{d} \frac{X_j^2(t)}{R(t)} = 1.$$

Levy's theorem implies that $B(t)$ is a Brownian motion. To see that CIR SDE holds note

$$\sqrt{R(t)}\mathrm{d}B(t) = \sum_{j=1}^{d} X_j(t)\mathrm{d}W_j(t)$$

Moreover,

$$\begin{aligned}
\mathrm{d}X_j^2(t) &= 2X_j(t)\mathrm{d}X_j(t) + \underbrace{\mathrm{d}X_j(t)\mathrm{d}X_j(t)}_{=\frac{\sigma^2}{4}\mathrm{d}t} \\
&= 2X_j(t)\left[-\tfrac{b}{2}X_j(t)\mathrm{d}t + \tfrac{1}{2}\sigma\mathrm{d}W_j(t)\right] + \frac{\sigma^2}{4}\mathrm{d}t \\
&= \left(\frac{\sigma^2}{4} - bX_j^2(t)\right)\mathrm{d}t + \sigma X_j(t)\mathrm{d}W_j(t)
\end{aligned}$$

Therefore,

$$\begin{aligned}
(a - bR(t))\,\mathrm{d}t + \sigma\sqrt{R(t)}\mathrm{d}B(t) &= \sum_{j=1}^{d}\left(\frac{\sigma^2}{4} - bX_j^2(t)\right)\mathrm{d}t + \sigma X_j(t)\mathrm{d}W_j(t) \\
&= \sum_{j=1}^{d} \mathrm{d}X_j^2(t) \\
&= \mathrm{d}R(t)
\end{aligned}$$

3. Since Itô integral with deterministic integrands are normally distributed at any fixed time t, $X_j(t)$ is normally distributed. Let $M_i(t) := \frac{2(X_i(t)-\mu_i(t))}{\sigma}$. Then $M_i(0) = 0$ and

$$\begin{aligned}
\mathrm{d}M_i(t)\mathrm{d}M_j(t) &= \frac{4}{\sigma^2}\mathrm{d}X_i(t)\mathrm{d}X_j(t) \\
&= \frac{4}{\sigma^2}\left(-\tfrac{b}{2}X_i(t)\mathrm{d}t + \tfrac{1}{2}\sigma\mathrm{d}W_i(t)\right)\cdot\left(-\tfrac{b}{2}X_j(t)\mathrm{d}t + \tfrac{1}{2}\sigma\mathrm{d}W_j(t)\right) \\
&= \delta_{ij}\mathrm{d}t.
\end{aligned}$$

Levy's theorem implies that $M_i(t)$ are independent Brownian motions. In particular, for each fix t, $M_i(t)$ and $M_j(t)$ are independent. Since $\mu_i(t)$ only depends on t, $X_i(t)$ and $X_j(t)$ are also independent.

4. It suffices to show the following fact

$$\mathbb{E}e^{uX^2} = \frac{1}{\sqrt{1-2u\sigma^2}} \exp\left(\frac{\mu^2 u}{1-2u\sigma^2}\right) \quad \text{for } u\sigma^2 < \frac{1}{2} \text{ where } X \sim \mathcal{N}(\mu, \sigma^2).$$

Letting $\gamma = \sqrt{1-2u\sigma^2}$,

$$\begin{aligned}
\mathbb{E}e^{uX^2} &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{uz^2 - \frac{(z-\mu)^2}{2\sigma^2}} \mathrm{d}z \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{uz^2 - \frac{(z-\mu)^2}{2\sigma^2}} \mathrm{d}z \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{\frac{2u\sigma^2 z^2 - (z-\mu)^2}{2\sigma^2}} \mathrm{d}z \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{-\frac{(1-2u\sigma^2)z^2 + 2\mu z - \mu^2}{2\sigma^2}} \mathrm{d}z \\
&= \frac{e^{\frac{\mu^2}{\gamma^2} - \mu^2}}{\gamma} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{-\frac{w^2 + \frac{2\mu}{\gamma} w - \frac{\mu^2}{\gamma^2}}{2\sigma^2}} \mathrm{d}w \\
&= \frac{e^{\frac{\mu^2}{\sigma^2\gamma^2} - \frac{\mu^2}{\sigma^2}}}{\gamma} \\
&= \frac{e^{\frac{\mu^2(1-\gamma^2)}{\sigma^2\gamma^2}}}{\gamma} \\
&= \frac{1}{\sqrt{1-2u\sigma^2}} \exp\left(\frac{\mu^2 u}{1-2u\sigma^2}\right).
\end{aligned}$$

Here we used the change of variables $w = \gamma z$. Since $\gamma > 0$ and z ranges from $-\infty$ and $+\infty$, it follows that w also ranges from $-\infty$ and $+\infty$.