## Exercise 6.7 (Heston stochastic volatility model)

Suppose that the stock price follows

$$\mathrm{d}S(t) = rS(t)\mathrm{d}t + \sqrt{V(t)}S(t)\mathrm{d}\tilde{W}_1(t)$$

Also the volatility term is governed by

$$\mathrm{d}V(t) = (a - bV(t))\mathrm{d}t + \sigma\sqrt{V(t)}\mathrm{d}\tilde{W}_2(t).$$

Here $a, b, \sigma > 0$ are constant. Moreover, $\tilde{W}_1(t)$ and $\tilde{W}_2(t)$ are correlated Brownian motions under $\tilde{\mathbb{P}}$ such that

$$\mathrm{d}\tilde{W}_1(t)\mathrm{d}\tilde{W}_2(t) = \rho t \text{ for some } \rho \in (-1, 1).$$

Denote by $c(t, s, v)$ price of a European call expiring at $T$ with strike price $K$. By Markov property, we have that

$$c(t, S(t), V(t)) = \tilde{\mathbb{E}}\left[e^{-r(T-t)}\left(S(T) - K\right)^+ |\mathcal{F}(t)\right]$$

In the region $t \in [0, T], s, v \geq 0$, show that

$$c_t + rsc_s + (a - bv)c_v + \frac{1}{2}s^2vc_{ss} + \rho\sigma svc_{sv} + \frac{1}{2}\sigma^2vc_{vv} = rc. \tag{1}$$

Moreover, prove that the following boundary condition holds.

$$c(T, s, v) = (s - K)^+ \text{ for all } s, v \geq 0.$$

### Proof

Iterated conditioning shows that $g(t, S(t), V(t)) = e^{-rt}c(t, S(t), V(t))$ is a martingale. Computing differentials while omitting the argument $(t, S(t), V(t))$ gives

$$\begin{aligned}
\mathrm{d}g(t, S(t), V(t)) &= g_t\mathrm{d}t + g_s\mathrm{d}S + g_v\mathrm{d}V + \frac{1}{2}g_{ss}\mathrm{d}S\mathrm{d}S + \frac{1}{2}g_{vv}\mathrm{d}V\mathrm{d}V + g_{sv}\mathrm{d}S\mathrm{d}V \\
&= g_t\mathrm{d}t + g_s\mathrm{d}S + g_v\mathrm{d}V + \frac{1}{2}g_{ss}vs^2\mathrm{d}t + \frac{1}{2}g_{vv}\sigma^2v\mathrm{d}t + g_{sv}\sigma\rho VS\mathrm{d}t \\
&= \left[g_t + \frac{1}{2}g_{ss}vs^2 + \frac{1}{2}g_{vv}\sigma^2v + g_{sv}\sigma\rho vs\right]\mathrm{d}t + g_s\left[rs\mathrm{d}t + \sqrt{v}s\mathrm{d}\tilde{W}_1\right] \\
&+ g_v\left[(a - bv)\mathrm{d}t + \sigma\sqrt{v}\mathrm{d}\tilde{W}_2\right] \\
&= \left[g_t + \frac{1}{2}g_{ss}vs^2 + \frac{1}{2}g_{vv}\sigma^2v + g_{sv}\sigma\rho vs + g_v(a - bv) + g_srs\right]\mathrm{d}t + g_s\sqrt{v}s\mathrm{d}\tilde{W}_1 + g_v\sigma\sqrt{v}\mathrm{d}\tilde{W}_2
\end{aligned}$$

The net $\mathrm{d}t$ term is zero as $g$ is a martingale. Thus,

$$g_t + \frac{1}{2}g_{ss}vs^2 + \frac{1}{2}g_{vv}\sigma^2v + g_{sv}\sigma\rho vs + g_v(a - bv) + g_srs = 0.$$

Reformulating in terms of $c(t, s, v)$, we get

$$e^{-rt}\left(-rc + c_t + \frac{1}{2}c_{ss}vs^2 + \frac{1}{2}c_{vv}\sigma^2v + c_{sv}\sigma\rho vs + c_v(a - bv) + c_srs\right) = 0$$

We immediately obtain the desired equation for $c$. To show the boundary condition, we follow the steps below.

- Suppose that $f(t,x,v)$ and $g(t,x,v)$, in the region $t \in [0,T]$, $x \in \mathbb{R}$ and $v \in \mathbb{R}^{\geq 0}$, satisfy

$$f_t + \left(r + \frac{v}{2}\right) f_x + (a - bv + \rho\sigma v)\, f_v + \frac{v}{2} f_{xx} + \rho\sigma v f_{xv} + \frac{\sigma^2 v}{2} f_{vv} = 0. \tag{2}$$

$$g_t + \left(r - \frac{v}{2}\right) g_x + (a - bv)\, g_v + \frac{v}{2} g_{xx} + \rho\sigma v g_{xv} + \frac{\sigma^2 v}{2} g_{vv} = 0. \tag{3}$$

We now show that the following function

$$c(t,s,v) = sf(t,\log s, v) - e^{-r(T-t)} K g(t,\log s, v).$$

satisfies (1). Omitting the argument $(t,\log s, v)$, we thus have that

$$\begin{aligned}
c_t &= sf_t - re^{-r(T-t)}Kg - e^{-r(T-t)}Kg_t \\
c_s &= f + f_s - e^{-r(T-t)}Ks^{-1}g_s \\
c_v &= sf_v - e^{-r(T-t)}Kg_v \\
c_{ss} &= s^{-1}f_s + s^{-1}f_{ss} - e^{-r(T-t)}Ks^{-2}g_{ss} + e^{-r(T-t)}Ks^{-2}g_s \\
c_{sv} &= f_v + f_{sv} - e^{-r(T-t)}Ks^{-1}g_{sv} \\
c_{vv} &= sf_{vv} - e^{-r(T-t)}Kg_{vv}
\end{aligned}$$

Omitting the argument $(t,\log s, v)$, we thus have that

$$\begin{aligned}
f\text{-term inside (1)} &= sf_t + rs[f + f_s] + (a - bv)sf_v + \tfrac{1}{2}svf_s + \tfrac{1}{2}svf_{ss} + \rho\sigma sv[f_v + f_{sv}] + \frac{1}{2}\sigma^2 vsf_{vv} \\
&= s[f_t + \left(r + \frac{v}{2}\right) f_s + (a - bv + \rho\sigma v)\, f_v + \frac{v}{2} f_{ss} + \rho\sigma v f_{sv} + \frac{\sigma^2 v}{2} f_{vv}] + rsf \\
&= rsf.
\end{aligned}$$

Similarly,

$$\begin{aligned}
g\text{-term inside (1)} &= -e^{-r(T-t)}K[rg + g_t + rg_s + (a - bv)g_v + \frac{v}{2}g_{ss} - \frac{v}{2}g_s + \rho\sigma v g_{sv} + \frac{\sigma^2 v}{2} g_{vv}] \\
&= -re^{-r(T-t)}Kg
\end{aligned}$$

Therefore,

$$\text{LHS in (1)} = rsf - re^{-r(T-t)}Kg = rc = \text{RHS in (1)}$$

- In the next step, we construct functions $f$ and $g$ satisfying (2) and (3) respectively. We start with $f$. Suppose that $X(t)$ and $V(t)$ satisfy the following:

$$\mathrm{d}X(t) = \left(r + \frac{1}{2}V(t)\right)\mathrm{d}t + \sqrt{V(t)}\mathrm{d}W_1(t)$$

$$\mathrm{d}V(t) = (a - bV(t) + \rho\sigma V(t))\,\mathrm{d}t + \sigma\sqrt{V(t)}\mathrm{d}W_2(t)$$

$W_1(t)$ and $W_2(t)$ are Brownian motion under some probability measure $\mathbb{P}$. Also holds that

$$\mathrm{d}W_1(t)\mathrm{d}W_2(t) = \rho t.$$

Define

$$f(t,x,v) = \mathbb{E}^{t,x,v}\mathbf{1}_{\{X(T)\geq \log K\}}$$

Note that the following boundary condition clearly holds.

$$f(T,x,v) = \mathbf{1}_{\{x\geq \log K\}} \text{ for all } x\in\mathbb{R}, v\geq 0.$$

Multidimensional Feynman-Kac gives

$$f_t + (r+\frac{1}{2}v)f_x + (a - bv + \rho\sigma v)\, f_v + \frac{v}{2}f_{xx} + \frac{\sigma^2 v}{2}f_{vv} + \sigma\rho v f_{xv} = 0.$$

Next, we construct function $g$. Consider

$$\mathrm{d}X(t) = \left(r - \frac{1}{2}V(t)\right)\mathrm{d}t + \sqrt{V(t)}\mathrm{d}W_1(t)$$

$$\mathrm{d}V(t) = (a - bV(t))\,\mathrm{d}t + \sigma\sqrt{V(t)}\mathrm{d}W_2(t)$$

Also holds that

$$\mathrm{d}W_1(t)\mathrm{d}W_2(t) = \rho t.$$

Define

$$g(t,x,v) = \mathbb{E}^{t,x,v}\mathbf{1}_{\{X(T)\geq \log K\}}$$

Note that the following boundary condition clearly holds.

$$g(T,x,v) = \mathbf{1}_{\{x\geq \log K\}} \text{ for all } x\in\mathbb{R}, v\geq 0.$$

Multidimensional Feynman-Kac gives

$$g_t + (r-\frac{1}{2}v)g_x + (a - bv)\, g_v + \frac{v}{2}g_{xx} + \frac{\sigma^2 v}{2}g_{vv} + \sigma\rho v g_{xv} = 0.$$

It remains to show the boundary condition for $c$. We have that

$$\begin{aligned} c(T,s,v) &= sf(T,\log s,v) - Kg(T,\log s,v) \\ &= s\mathbf{1}_{\{\log s\geq \log K\}} - K\mathbf{1}_{\{\log s\geq \log K\}} \\ &= (s-k)\cdot\mathbf{1}_{\{s\geq K\}} \\ &= (s-K)^+. \end{aligned}$$

Proof is complete.