Exercise 6.8 (Kolmogrov backward equation)

Consider the following SDE

$$\mathrm{d}X(u) = \beta(u, X(u))\mathrm{d}u + \gamma(u, X(u))\mathrm{d}W(u) \text{ where } X(t) = x.$$

Denote by $p(t, T, x, y)$ the transition density for the solution to this equation. In other words,

$$g(t, x) = \mathbb{E}^{t,x} h(X(T)) = \int_0^{+\infty} h(y) p(t, T, x, y)\mathrm{d}y.$$

We assume that $p(t, T, x, y) = 0$ for $0 \leq t < T$ and $y \leq 0$. Show that the following PDE holds:

$$-p_t(t, T, x, y) = \beta(t, x) p_x(t, T, x, y) + \tfrac{1}{2}\gamma^2(t, x) p_{xx}(t, T, x, y)$$

Proof

Recall that the following PDE holds

$$g_t(t, x) + \beta(t, x) g_x(t, x) + \tfrac{1}{2}\gamma^2(t, x) g_{xx}(t, x) = 0$$

We therefore have that

$$\int_0^{+\infty} h(y) \left[p_t(t, T, x, y) + \beta(t, x) p_x(t, T, x, y) + \tfrac{1}{2}\gamma^2(t, x) p_{xx}(t, T, x, y) \right] \mathrm{d}y = 0.$$

This is equation holds for any Borel-measurable function $h(y)$, we must have that

$$p_t(t, T, x, y) + \beta(t, x) p_x(t, T, x, y) + \tfrac{1}{2}\gamma^2(t, x) p_{xx}(t, T, x, y) = 0.$$