Exercise 6.9 (Kolmogorov forward equation)

Consider the following SDE

$$\mathrm{d}X(u) = \beta(u, X(u))\mathrm{d}u + \gamma(u, X(u))\mathrm{d}W(u) \text{ where } X(t) = x.$$

Denote by $p(t, T, x, y)$ the transition density for the solution to this equation. In other words,

$$g(t, x) = \mathbb{E}^{t,x} h(X(T)) = \int_0^{+\infty} h(y)p(t, T, x, y)\mathrm{d}y.$$

We assume that $p(t, T, x, y) = 0$ for $0 \leq t < T$ and $y \leq 0$. Show that $p(t, T, x, y)$ satisfies the following equation

$$\frac{\partial}{\partial T}p(t, T, x, y) = -\frac{\partial}{\partial y}\left(\beta(t, y)p(t, T, x, y)\right) + \tfrac{1}{2}\frac{\partial^2}{\partial y^2}\left(\gamma^2(T, y)p(t, T, x, y)\right)$$

Proof

Denote by

$$M(t, T, x, y) = \frac{\partial}{\partial T}p(t, T, x, y) + \frac{\partial}{\partial y}\left(\beta(t, y)p(t, T, x, y)\right) - \tfrac{1}{2}\frac{\partial^2}{\partial y^2}\left(\gamma^2(T, y)p(t, T, x, y)\right)$$

By assumption $M(t, T, x, y)$ is a continuous function of y. Therefore, for fixed t and T, if $M \not\equiv 0$, then there exists $0 < a < b$ such that $M(t, T, x, y)$ is strictly positive or strictly negative for $y \in (a, b)$. Consequently, if h is a smooth function such that

- $h(y) = 0$ for all $y \notin (a, b)$
- $h'(0) = h'(b) = 0$
- $h(y) > 0$ for all $y \in (a, b)$,

then it must hold that

$$\int_0^b h(y)M(t, T, x, y)\mathrm{d}y \neq 0.$$

It is not difficult to verify that such function h exists. For example, define

$$\alpha(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

It is easy to check that α is a smooth function. Then $\alpha(1 - x)\alpha(1 + x)$ is a smooth function which is identically zero outside $(-1, 1)$ and positive inside $(-1, 1)$. Itô formula gives

$$\begin{aligned} \mathrm{d}h(X(u)) &= h'(X(u))\mathrm{d}X(u) + \tfrac{1}{2}h''(X(u))\mathrm{d}X(u)\mathrm{d}X(u) \\ &= h'(X(u))\beta(u, X(u))\mathrm{d}u + h'(X(u))\gamma(u, X(u))\mathrm{d}W(u) + \tfrac{1}{2}h''(X(u))\gamma^2(u, X(u))\mathrm{d}u \end{aligned}$$

Integrating both sides from t to T gives

$$h(X(T)) = h(x) + \int_t^T \left(h'(X(u))\beta(u, X(u)) + \tfrac{1}{2}h''(X(u))\gamma^2(u, X(u))\right)\mathrm{d}u + \text{ Itô integral}$$

Since $X(u)$ has density $p(t,u,x,y)$ in the y-variable, taking expectation from both sides gives

$$\int_0^b h(y)p(t,T,x,y)\mathrm{d}y = h(x) + \int_t^T \int_0^b \left(h'(y)\beta(u,y)p(t,u,x,y) + \tfrac{1}{2}h''(y)\gamma^2(u,y)p(t,u,x,y)\right) \mathrm{d}y\mathrm{d}u.$$

On the other hand, integration by parts gives

$$\int_0^b h'(y)\beta(u,y)p(t,u,x,y)\mathrm{d}y + \int_0^b h(y)\frac{\partial}{\partial y}[\beta(u,y)p(t,u,x,y)]\mathrm{d}y = h(y)\beta(u,y)p(t,u,x,y)|_0^b = 0.$$

Here we used the fact that $h(0) = h(b) = 0$. Similarly since $h'(0) = h'(b) = 0$, we have that

$$\int_0^b h''(y)\gamma^2(u,y)p(t,u,x,y)\mathrm{d}y + \int_0^b h'(y)\frac{\partial}{\partial y}[\gamma^2(u,y)p(t,u,x,y)]\mathrm{d}y = h'(y)\gamma^2(u,y)p(t,u,x,y)|_0^b = 0.$$

Another integration by parts gives

$$\int_0^b h'(y)\frac{\partial}{\partial y}[\gamma^2(u,y)p(t,u,x,y)]\mathrm{d}y + \int_0^b h(y)\frac{\partial^2}{\partial y^2}[\gamma^2(u,y)p(t,u,x,y)]\mathrm{d}y = h(y)\frac{\partial}{\partial y}[\gamma^2(u,y)p(t,u,x,y)]|_0^b = 0.$$

We thus have shown that

$$\int_0^b h''(y)\gamma^2(u,y)p(t,u,x,y)\mathrm{d}y = \int_0^b h(y)\frac{\partial^2}{\partial y^2}[\gamma^2(u,y)p(t,u,x,y)]\mathrm{d}y.$$

Thus, we have that

$$\int_0^b h(y)p(t,T,x,y)\mathrm{d}y = h(x) + \int_t^T \int_0^b h(y)\left(-\frac{\partial}{\partial y}[\beta(u,y)p(t,u,x,y)] + \frac{\partial^2}{\partial y^2}[\gamma^2(u,y)p(t,u,x,y)]\right)\mathrm{d}y\mathrm{d}u.$$

From elementary calculus $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{g(x)} f(t)\mathrm{d}t = f(g(x))) \cdot g'(x)$. Differentiate both sides w.r.t T to obtain

$$\int_0^b h(y)\frac{\partial}{\partial T}p(t,T,x,y)\mathrm{d}y = \int_0^b h(y)\left(-\frac{\partial}{\partial y}[\beta(T,y)p(t,T,x,y)] + \frac{\partial^2}{\partial y^2}[\gamma^2(T,y)p(t,T,x,y)]\right)\mathrm{d}y$$

Rearranging gives

$$\int_0^b h(y)M(t,T,x,y)\mathrm{d}y = 0.$$

This is the desired contradiction. As h is positive on (a,b) and zero elsewhere; But, by assumption, $M(t,T,x,y)$ is strictly positive for every $y \in (a,b)$ or strictly negative for every $y \in (a,b)$.