Exercise 7.2 (Boundary conditions for the up-and-out call)

Closed-form formula for the up-and-out call option from previous exercise is calculated as below. Set

$$\delta_\pm(\tau, s) = \frac{1}{\sigma\sqrt{\tau}}\left[\log s + \left(r \pm \tfrac{1}{2}\sigma^2\right)\tau\right]$$

Then

$$\begin{aligned} v(t,x) &= x\left[N\left(\delta_+\left(\tau, \frac{x}{K}\right)\right) - N\left(\delta_+\left(\tau, \frac{x}{B}\right)\right)\right] \\ &\quad - e^{-r\tau}K\left[N\left(\delta_-\left(\tau, \frac{x}{K}\right)\right) - N\left(\delta_-\left(\tau, \frac{x}{B}\right)\right)\right] \\ &\quad - B\left(\frac{x}{B}\right)^{-\frac{2r}{\sigma^2}}\left[N\left(\delta_+\left(\tau, \frac{B^2}{Kx}\right)\right) - N\left(\delta_+\left(\tau, \frac{B}{x}\right)\right)\right] \\ &\quad + e^{-r\tau}K\left(\frac{x}{B}\right)^{-\frac{2r}{\sigma^2}+1}\left[N\left(\delta_-\left(\tau, \frac{B^2}{Kx}\right)\right) - N\left(\delta_-\left(\tau, \frac{B}{x}\right)\right)\right]. \end{aligned}$$

In this exercise, we show that

- **Boundary Condition I:** $v(t,0) = 0, 0 \le t \le T$
- **Boundary Condition II:** $v(t,B) = 0, 0 \le t < T$
- **Boundary Condition III:** $v(T,x) = (x-K)^+, 0 \le x < B$

Remark: This exercise does not show the boundary condition $v(T,B) = B-K$. Remember that v is discontinuous at (T,B), but it is continuous elsewhere inside $\{(t,x) : 0 \le t \le T, 0 \le x \le B\}$. It is emphasized that $v(t,x)$ is defined for $\tau = 0$, or $x = 0, B$ thanks to these boundary conditions.

Proof

We begin by noting that for $\tau > 0$

$$\begin{aligned}
v(t,B) &= B\left[N\left(\delta_+\left(\tau,\frac{B}{K}\right)\right) - N\left(\delta_+\left(\tau,1\right)\right)\right] \\
&- e^{-r\tau}K\left[N\left(\delta_-\left(\tau,\frac{B}{K}\right)\right) - N\left(\delta_-\left(\tau,1\right)\right)\right] \\
&- B\left[N\left(\delta_+\left(\tau,\frac{B}{K}\right)\right) - N\left(\delta_+\left(\tau,1\right)\right)\right] \\
&+ e^{-r\tau}K\left[N\left(\delta_-\left(\tau,\frac{B}{K}\right)\right) - N\left(\delta_-\left(\tau,1\right)\right)\right] \\
&= B\left[N\left(\delta_+\left(\tau,\frac{B}{K}\right)\right) - N\left(\delta_+\left(\tau,1\right)\right)\right] \\
&- B\left[N\left(\delta_+\left(\tau,\frac{B}{K}\right)\right) - N\left(\delta_+\left(\tau,1\right)\right)\right] \\
&- e^{-r\tau}K\left[N\left(\delta_-\left(\tau,\frac{B}{K}\right)\right) - N\left(\delta_-\left(\tau,1\right)\right)\right] \\
&+ e^{-r\tau}K\left[N\left(\delta_-\left(\tau,\frac{B}{K}\right)\right) - N\left(\delta_-\left(\tau,1\right)\right)\right] \\
&= 0.
\end{aligned}$$

Boundary condition II thus holds. Next, we show boundary conditions I and III respectively. To show condition I, it is fine to assume $\tau > 0$ as the case $\tau = 0$ will be considered in boundary condition III. As $x \to 0$, it must hold that

$$\delta_\pm(\tau,\frac{x}{K}), \delta_\pm(\tau,\frac{x}{B}) \to -\infty.$$

Therefore,

$$\begin{aligned}
N\left(\delta_+\left(\tau,\frac{x}{K}\right)\right) - N\left(\delta_+\left(\tau,\frac{x}{B}\right)\right) \to 0 \\
N\left(\delta_-\left(\tau,\frac{x}{K}\right)\right) - N\left(\delta_-\left(\tau,\frac{x}{B}\right)\right) \to 0
\end{aligned}$$

Let $\delta \in \{-1,1\}$. Fix a constant c. There exists constants c_1, c_2 such that

$$\delta_\pm\left(\tau, cx^\delta\right) = c_1 \log x + c_2$$

Thus, for $p > 0$,

$$\begin{aligned}
\lim_{x\downarrow 0} \frac{N\left(\delta_\pm\left(\tau,cx^\delta\right)\right)}{x^p} &= \lim_{x\downarrow 0} \frac{\exp\left(-\frac{1}{2}\delta_\pm^2\left(\tau,cx^\delta\right)\right)\cdot\frac{c_1}{x}}{px^{p-1}} \\
&= \lim_{x\downarrow 0} c_3 \frac{\exp\left(-\frac{1}{2}\delta_\pm^2\left(\tau,cx^\delta\right)\right)}{x^p}
\end{aligned}$$

Here c_3 is a constant. Continuing,

$$\begin{aligned}\exp\left(-\frac{1}{2}\delta_{\pm}^2\left(\tau, cx^{\delta}\right)\right) &= \exp\left(-\lambda_2 \log^2 x + \lambda_1 \log x + \lambda_0\right)\\ &= \exp\left(-\lambda_2\left[\log x + \lambda_3\right]^2 + \lambda_4\right)\\ &= \exp\left(-\lambda_2\left[\log e^{\lambda_3} x\right]^2 + \lambda_4\right)\\ &= \exp\left(-\lambda_2 \mu^2 + \lambda_4\right)\end{aligned}$$

Here λ_i are constant and more so $\lambda_2 > 0$. Moreover, $x = e^{\mu - \lambda_3}$. Thus,

$$\lim_{x\downarrow 0} \frac{\exp\left(-\frac{1}{2}\delta_{\pm}^2\left(\tau, cx^{\delta}\right)\right)}{x^p} = e^{\lambda_4} \cdot \lim_{\mu\downarrow -\infty} \frac{1}{e^{\lambda_2\mu^2 + p(\mu - \lambda_3)}} = 0$$

Last equality follows since $\lambda_2 > 0$. In conclusion, as $x \to 0$

$$\begin{aligned}-B\left(\frac{x}{B}\right)^{-\frac{2r}{\sigma^2}}\left[N\left(\delta_+\left(\tau, \frac{B^2}{Kx}\right)\right) - N\left(\delta_+\left(\tau, \frac{B}{x}\right)\right)\right] &\to 0\\ e^{-r\tau}K\left(\frac{x}{B}\right)^{-\frac{2r}{\sigma^2}+1}\left[N\left(\delta_-\left(\tau, \frac{B^2}{Kx}\right)\right) - N\left(\delta_-\left(\tau, \frac{B}{x}\right)\right)\right] &\to 0\end{aligned}$$

Putting pieces together, boundary condition I holds. It remains to show boundary condition III. First, note that for $c > 0$,

$$\lim_{\tau\downarrow 0} \delta_{\pm}(\tau, c) = \begin{cases} -\infty & \text{if } 0 < c < 1\\ 0 & \text{if } c = 1\\ +\infty & \text{if } c > 1.\end{cases}$$

By assumption, $K < B$ as otherwise the option needs to cross the barrier to end up in the money. We consider the following cases.

$x < K$. In this case,

$$\begin{aligned}\lim_{\tau\downarrow 0} N\left(\delta_{\pm}\left(\tau, \frac{x}{K}\right)\right) &\to 0\\ \lim_{\tau\downarrow 0} N\left(\delta_{\pm}\left(\tau, \frac{x}{B}\right)\right) &\to 0\\ \lim_{\tau\downarrow 0} N\left(\delta_{\pm}\left(\tau, \frac{B^2}{Kx}\right)\right) &\to 1\\ \lim_{\tau\downarrow 0} N\left(\delta_{\pm}\left(\tau, \frac{B}{x}\right)\right) &\to 1\end{aligned}$$

Thus, $v(T, x) = (x - K)^+$ holds in this case.

$x = K$. In this case,

$$\lim_{\tau\downarrow 0} N\left(\delta_\pm\left(\tau, \frac{x}{K}\right)\right) \to N(0)$$
$$\lim_{\tau\downarrow 0} N\left(\delta_\pm\left(\tau, \frac{x}{B}\right)\right) \to 0$$
$$\lim_{\tau\downarrow 0} N\left(\delta_\pm\left(\tau, \frac{B^2}{Kx}\right)\right) \to 1$$
$$\lim_{\tau\downarrow 0} N\left(\delta_\pm\left(\tau, \frac{B}{x}\right)\right) \to 1$$

Thus, in this case

$$v(T, x) = xN(0) - KN(0) = KN(0) - KN(0) = 0.$$

$K < x < B$. In this case,

$$\lim_{\tau\downarrow 0} N\left(\delta_\pm\left(\tau, \frac{x}{K}\right)\right) \to 1$$
$$\lim_{\tau\downarrow 0} N\left(\delta_\pm\left(\tau, \frac{x}{B}\right)\right) \to 0$$
$$\lim_{\tau\downarrow 0} N\left(\delta_\pm\left(\tau, \frac{B^2}{Kx}\right)\right) \to 1$$
$$\lim_{\tau\downarrow 0} N\left(\delta_\pm\left(\tau, \frac{B}{x}\right)\right) \to 1$$

Thus,

$$\begin{aligned} v(T, x) &= \lim_{\tau\downarrow 0} v(t, x) \\ &= \lim_{\tau\downarrow 0} x - e^{-r\tau}K \\ &= x - K. \end{aligned}$$