Exercise 7.3 (Markov property of GBM & its max-to-date process)

Consider a geometric Brownian motion and its max-date process $Y(t) := \max_{0 \le u \le t} S(u)$. Use Independence Lemma to show that for $t \in [0, T]$ and any function $f(s, y)$, there exists another function $g(s, y)$ such that

$$\mathbb{E}[f(S(T), Y(T))|\mathcal{F}(t)] = g(S(t), Y(t)) \tag{1}$$

Proof

For $u \ge t$,

$$S(u) = S(t) \cdot \underbrace{\exp\left(\sigma \cdot (W(u) - W(t)) + \left(\alpha - \frac{\sigma^2}{2}\right)(u - t)\right)}_{:=Q_t(u)}$$

Note that $Q_t(u)$ is independent of $\mathcal{F}(t)$. Moreover,

$$Y(T) = \max\{Y(t), S(t) \cdot \max_{t \le u \le T} Q_t(u)\}.$$

Write

$$h(s, y, q_T, q) := f(s \cdot q_T, \max\{y, s \cdot q\})$$

Then

$$\mathbb{E}[f(S(T), Y(T))|\mathcal{F}(t)] = \mathbb{E}[h(S(t), Y(t), Q_T(u), \max_{t \le u \le T} Q_t(u))|\mathcal{F}(t)]$$

Since $S(t), Y(t)$ are $\mathcal{F}(t)$-measurable and $\max_{t \le u \le T} Q_t(u)$ is independent of it, Eq (1) holds where

$$g(s, y) := \mathbb{E}h(s, y, Q_T(u), \max_{t \le u \le T} Q_t(u)).$$