Exercise 7.4 (Cross variation of GBM & its max-to-date process)

Let $S(t)$ be a geometric Brownian motion and fix $T > 0$. Denote by $Y(t)$ its max-to-date process. Prove that

$$\sum_{j=1}^{m} (Y(t_j) - Y(t_{j-1})) \cdot (S(t_j) - S(t_{j-1})) \to 0 \text{ whenever } t_j - t_{j-1} \to 0 \ \& \ m \to +\infty.$$

Proof

Letting $\epsilon := \max_j |S(t_j) - S(t_{j-1})|$,

$$\begin{aligned}\left|\sum_{j=1}^{m} (Y(t_j) - Y(t_{j-1})) \cdot (S(t_j) - S(t_{j-1}))\right| &\le \sum_{j} (Y(t_j) - Y(t_{j-1})) \cdot |S(t_j) - S(t_{j-1})| \\ &\le \epsilon \cdot \sum_{j} Y(t_j) - Y(t_{j-1}) \\ &\le \epsilon \cdot (Y(T) - Y(0)) \,.\end{aligned}$$

Since $S(t)$ is continuous, the result immediately follows.