Exercise 7.6 (Boundary conditions for lookback option)

The dimensionally reduced, two-variables pricing function for lookback options is as below:

$$u(t,z) = \left(1+\frac{\sigma^2}{2r}\right) zN(\delta_+(\tau,z)) + e^{-r\tau}N(-\delta_-(\tau,z))$$
$$-\frac{\sigma^2}{2r}e^{-r\tau}z^{1-\frac{2r}{\sigma^2}}N(-\delta_-(\tau,z^{-1})) - z$$

Here $0 \le t < T$, $0 < z \le 1$. In this exercise, we show two boundary conditions for $u(t,z)$:

Boundary condition I: $u(t,0) = e^{-r\tau}$ for $0 \le t < T$

Boundary condition II: $u(T,z) = 1 - z$ for $0 < z \le 1$

Proof

Fix $t \in [0.T)$ and let $z \downarrow 0$,

$$\delta_+(\tau,z) = \frac{1}{\sigma\sqrt{\tau}}\left[\log z + \left(r+\tfrac{1}{2}\sigma^2\right)\tau\right] \to -\infty$$
$$-\delta_-(\tau,z) = -\frac{1}{\sigma\sqrt{\tau}}\left[\log z + \left(r-\tfrac{1}{2}\sigma^2\right)\tau\right] \to +\infty$$
$$-\delta_-(\tau,z^{-1}) = -\frac{1}{\sigma\sqrt{\tau}}\left[-\log z + \left(r-\tfrac{1}{2}\sigma^2\right)\tau\right] \to -\infty$$

Therefore,

$$\lim_{z\downarrow 0} u(t,z) = \lim_{z\downarrow 0} \underbrace{\left(1+\frac{\sigma^2}{2r}\right) zN(\delta_+(\tau,z))}_{\to 0} + \underbrace{e^{-r\tau}N(-\delta_-(\tau,z))}_{\to e^{-r\tau}}$$
$$-\frac{\sigma^2}{2r}e^{-r\tau}z^{1-\frac{2r}{\sigma^2}}N(-\delta_-(\tau,z^{-1})) - \underbrace{z}_{\to 0}$$

It remains to show show that

$$\lim_{z\downarrow 0} z^{1-\frac{2r}{\sigma^2}}N(-\delta_-(\tau,z^{-1})) = 0$$

Let $p = \frac{2r}{\sigma^2} - 1$. If $p < 0$, then the above statement clearly holds. Assume $p > 0$. In the following chain, c stands for some constant. These constants are not necessary equal to each other. For $\lambda > 0$,

$$\delta_-^2(\tau,z^{-1}) = 2\lambda\log^2 z + c\log z + c$$

Continuing,

$$\begin{aligned}
\lim_{z\downarrow 0} z^{1-\frac{2r}{\sigma^2}} N(-\delta_-(\tau, z^{-1})) &= \lim_{z\downarrow 0} \frac{N(-\delta_-(\tau, z^{-1}))}{z^p} \\
&= \lim_{z\downarrow 0} \frac{\exp\left(-\frac{1}{2}\delta_-^2(\tau, z^{-1})\right)\cdot cz^{-1}}{pz^{p-1}} \\
&= c\lim_{z\downarrow 0} \frac{\exp\left(-\frac{1}{2}\delta_-^2(\tau, z^{-1})\right)}{z^p} \\
&= c\lim_{\mu\downarrow -\infty} \frac{\exp\left(-\lambda\mu^2 + c\mu + c\right)}{e^{\mu p}} \\
&= c\lim_{\mu\downarrow -\infty} e^{-\lambda\mu^2 + c\mu + c} \\
&= 0.
\end{aligned}$$

We thus have established boundary condition I. To show boundary condition II, we note

$$\lim_{\tau\downarrow 0} \delta_\pm(\tau, z) = \begin{cases} -\infty & \text{if } 0 < z < 1 \\ 0 & \text{if } z = 1 \\ +\infty & \text{if } z > 1 \end{cases}$$

Fix $z \in (0, 1)$. Then

$$\begin{aligned}
\lim_{\tau\downarrow 0} u(t, z) = \lim_{\tau\downarrow 0} &\underbrace{\left(1 + \frac{\sigma^2}{2r}\right) zN(\delta_+(\tau, z))}_{\to 0} + \underbrace{e^{-r\tau}N(-\delta_-(\tau, z))}_{\to 1} \\
&- \underbrace{\frac{\sigma^2}{2r} e^{-r\tau} z^{1-\frac{2r}{\sigma^2}} N(-\delta_-(\tau, z^{-1}))}_{\to 0} - z \\
&= 1 - z.
\end{aligned}$$

For $z = 1$, it holds that

$$\begin{aligned}
\lim_{\tau\downarrow 0} u(t, z) = \lim_{\tau\downarrow 0} &\left(1 + \frac{\sigma^2}{2r}\right) \underbrace{zN(\delta_+(\tau, z))}_{\to N(0)} + \underbrace{e^{-r\tau}N(-\delta_-(\tau, z))}_{\to N(0)} \\
&- \frac{\sigma^2}{2r} \underbrace{e^{-r\tau} z^{1-\frac{2r}{\sigma^2}} N(-\delta_-(\tau, z^{-1}))}_{\to N(0)} - \underbrace{z}_{z=1} \\
&= \left(1 + \frac{\sigma^2}{2r}\right) N(0) + N(0) - \frac{\sigma^2}{2r} N(0) - 1 \\
&= 2N(0) - 1 \\
&= 0.
\end{aligned}$$

Boundary condition II has also been proved.