Exercise 7.7 (Zero-strike Asian Call)

Consider a zero-strike Asian call with payoff at T

$$V(T) = \frac{1}{T}\int_0^T S(u)\mathrm{d}u$$

Construct a hedge for this option.

Proof

We begin by computing

$$\begin{aligned}
e^{-r(T-t)}\tilde{\mathbb{E}}\left[\frac{1}{T}\int_0^T S(u)\mathrm{d}u|\mathcal{F}(t)\right] &= \frac{1}{T}\int_0^T e^{ru-r(T-t)}\tilde{\mathbb{E}}\left[e^{-ru}S(u)|\mathcal{F}(t)\right]\mathrm{d}u \\
&= \frac{1}{T}\int_t^T e^{ru-r(T-t)}e^{-rt}S(t)\mathrm{d}u + e^{-r(T-t)}\tilde{\mathbb{E}}\left[\frac{1}{T}\int_0^t S(u)\mathrm{d}u|\mathcal{F}(t)\right] \\
&= \frac{x}{T}\int_t^T e^{ru-r(T-t)}e^{-rt}\mathrm{d}u + \frac{e^{-r(T-t)}y}{T} \\
&= \frac{xe^{-rT}}{T}\int_t^T e^{ru}\mathrm{d}u + \frac{e^{-r(T-t)}y}{T} \\
&= \frac{x}{rT}\cdot\left(1-e^{-r(T-t)}\right) + \frac{y}{T}\cdot e^{-r(T-t)}.
\end{aligned}$$

Thus,

$$v(t,x,y) = \frac{x}{rT}\cdot\left(1-e^{-r(T-t)}\right) + \frac{y}{T}\cdot e^{-r(T-t)}.$$

We will show that $v(t,x,y)$ satisfies BSM equation

$$v_t + rxv_x + xv_y + \frac{1}{2}\sigma^2x^2v_{xx} = rv.$$

We have that

$$\begin{aligned}
v_t &= -\frac{x}{T}\cdot e^{-r(T-t)} + \frac{ry}{T}\cdot e^{-r(T-t)} \\
v_x &= \frac{1}{rT}\cdot\left(1-e^{-r(T-t)}\right) \\
v_y &= \frac{1}{T}\cdot e^{-r(T-t)} \\
v_{xx} &= 0
\end{aligned}$$

Therefore,

$$\begin{aligned}
v_t + rxv_x + xv_y + \frac{1}{2}\sigma^2x^2v_{xx} &= -\frac{x}{T}\cdot e^{-r(T-t)} + \frac{ry}{T}\cdot e^{-r(T-t)} + \frac{x}{T}\cdot\left(1-e^{-r(T-t)}\right) + \frac{x}{T}\cdot e^{-r(T-t)} \\
&= \frac{x}{T}\cdot\left(1-e^{-r(T-t)}\right) + \frac{ry}{T}\cdot e^{-r(T-t)} \\
&= rv.
\end{aligned}$$

Moreover, the following boundary condition holds

- $v(T, x, y) = \frac{y}{T}$
- $v(t, 0, y) = \frac{y}{T} \cdot e^{-r(T-t)}$

Notice next that

$$\Delta(t) := v_x(t, S(t), Y(t)) = \frac{1}{rT} \cdot \left(1 - e^{-r(T-t)}\right)$$

is deterministic. Now consider a portfolio with initial capital

$$X(0) = v(0, S(0), 0) = \frac{S(0)}{rT} \cdot \left(1 - e^{-rT}\right).$$

where at each time t, the agent holds $\Delta(t)$ shares of stock. She may borrow or invest at interest rate r. Therefore,

$$\begin{aligned} \mathrm{d}X(t) &= \Delta(t)\mathrm{d}S(t) + r(X(t) - \Delta(t)S(t))\mathrm{d}t \\ &= rX(t)\mathrm{d}t + \Delta(t)\left(\mathrm{d}S(t) - rS(t)\mathrm{d}t\right) \end{aligned}$$

Thus, noting that $\mathrm{d}\Delta(t) = \frac{-e^{-r(T-t)}}{T}\mathrm{d}t$,

$$\begin{aligned} \mathrm{d}e^{r(T-t)}X(t) &= -re^{r(T-t)}X(t)\mathrm{d}t + e^{r(T-t)}\mathrm{d}X(t) \\ &= e^{r(T-t)}\left(\mathrm{d}X(t) - rX(t)\mathrm{d}t\right) \\ &= e^{r(T-t)}\Delta(t)\left(\mathrm{d}S(t) - rS(t)\mathrm{d}t\right) \\ &= \Delta(t)\mathrm{d}e^{r(T-t)}S(t) \\ &= \underbrace{\Delta(t)\mathrm{d}e^{r(T-t)}S(t) + e^{r(T-t)}S(t)\mathrm{d}\Delta(t)}_{=\mathrm{d}e^{r(T-t)}\Delta(t)S(t)} - \underbrace{e^{r(T-t)}S(t)\mathrm{d}\Delta(t)}_{=-\frac{1}{T}S(t)\mathrm{d}t} \\ &= \mathrm{d}e^{r(T-t)}\Delta(t)S(t) + \frac{1}{T}S(t)\mathrm{d}t. \end{aligned}$$

Continuing,

$$X(T) - e^{rT}X(0) = \underbrace{\Delta(T)}_{=0}S(T) - \underbrace{e^{rT}\Delta(0)}_{=\frac{1}{rT}\cdot(e^{rT}-1)}S(0) + \frac{1}{T}\int_0^T S(u)\mathrm{d}u$$

Thus,

$$\begin{aligned} X(T) &= \frac{S(0)}{rT} \cdot \left(e^{rT} - 1\right) - e^{rT}\Delta(0)S(0) + \frac{1}{T}\int_0^T S(u)\mathrm{d}u \\ &= \frac{1}{T}\int_0^T S(u)\mathrm{d}u \end{aligned}$$