Exercise 7.8

Consider continuously sampling Asian option where interest rate is 0. Denote by $\gamma(t)$ the number of shares in risky asset. Find non-random $X(0)$ and $\gamma(t)$ such that

$$X(T) = \tfrac{1}{c} \int_{T-c}^{T} S(u) \mathrm{d}u - K$$

Proof

We replicate the proof of the case $r > 0$ by letting $r \downarrow 0$. Notice that

$$\lim_{r \downarrow 0} \frac{1 - e^{-rc}}{rc} = \lim_{r \downarrow 0} \frac{ce^{-rc}}{c} = 1$$

Moreover,

$$\lim_{r \downarrow 0} \frac{1 - e^{-r(T-t)}}{rc} = \lim_{r \downarrow 0} \frac{(T-t)e^{-r(T-t)}}{c} = \frac{T-t}{c}.$$

We therefore define

$$\gamma(t) = \begin{cases} 1 & 0 \le t \le T-c, \\ \frac{T-t}{c} & T-c \le t \le T. \end{cases}$$

We also take the initial capital of

$$X(0) = S(0) - K.$$

$\gamma(t)$ is the number of shares of the risky asset held at time t. Therefore,

$$\begin{aligned} \mathrm{d}X(t) &= \gamma(t)\mathrm{d}S(t) + \underbrace{r}_{=0} (X(t) - \gamma(t)S(t))\, \mathrm{d}t \\ &= \gamma(t)\mathrm{d}S(t) \\ &= \mathrm{d}\gamma(t)S(t) - S(t)\mathrm{d}\gamma(t). \end{aligned}$$

Thus,

$$\mathrm{d}X(t) = \mathrm{d}\gamma(t)S(t) - S(t)\mathrm{d}\gamma(t). \tag{1}$$

Since γ is constant for $0 \le t \le T-c$, it holds that

$$X(t) = S(t) - K \quad \forall\, 0 \le t \le T-c$$

Here at time zero, we buy one share of stock and borrow K from the money market account. Since $r = 0$, for any time before $T-c$, we still owe K to money market and our stock is worth $S(t)$. Next, integration of (1) from $T-c$ to T gives

$$\begin{aligned} X(T) &= S(T-c) - K + \int_{T-c}^{T} \mathrm{d}\gamma(t)S(t) - \int_{T-c}^{T} S(t)\mathrm{d}\gamma(t) \\ &= S(T-c) - K + \underbrace{\gamma(T)}_{=0} S(T) - \underbrace{\gamma(T-c)}_{=1} S(T-c) + \frac{1}{c}\int_{T-c}^{T} S(t)\mathrm{d}t \\ &= \frac{1}{c}\int_{T-c}^{T} S(t)\mathrm{d}t - K. \end{aligned}$$