Exercise 7.9

Proof

- Note that

$$v(t,s,x) = sg(t, \underbrace{\frac{x}{s}}_{:=y})$$

A simple application of chain rule yields that

$$\begin{aligned}
v_t =& sg_t \\
v_s =& g - s\frac{x}{s^2}g_y = g - yg_y \\
v_x =& g_y \\
v_{ss} =& -\frac{x}{s^2}g_y + \frac{x}{s^2}g_y + \frac{y^2}{s}g_{yy} = \frac{y^2}{s}g_{yy} \\
v_{sx} =& \frac{1}{s}g_y - \frac{1}{s}g_y - \frac{y}{s}g_{yy} = -\frac{y}{s}g_{yy} \\
v_{xx} =& \frac{1}{s}g_{yy}
\end{aligned}$$

- We need to verify that $e^{-rt}v(t, S(t), X(t))$ is a martingale under $\tilde{\mathbb{P}}$. We have that

$$\mathrm{d}e^{-rt}v = -re^{-rt}v\mathrm{d}t + e^{-rt}\mathrm{d}v$$

Continuing,

$$\mathrm{d}v = v_t\mathrm{d}t + v_s\mathrm{d}S + v_x\mathrm{d}X + \tfrac{1}{2}v_{ss}\mathrm{d}S\mathrm{d}S + \tfrac{1}{2}v_{xx}\mathrm{d}X\mathrm{d}X + v_{sx}\mathrm{d}S\mathrm{d}X$$

Remember,

$$\mathrm{d}S = rS\mathrm{d}t + \sigma S\mathrm{d}\tilde{W}.$$

Moreover,

$$\mathrm{d}X = rX\mathrm{d}t + \sigma\gamma S\mathrm{d}\tilde{W}.$$

Therefore,

$$\begin{aligned}
\mathrm{d}v &= \left[v_t + rsv_s + rxv_x + \tfrac{1}{2}\sigma^2 s^2 v_{ss} + \tfrac{1}{2}\sigma^2\gamma^2 s^2 v_{xx} + \sigma^2 s^2\gamma v_{sx}\right]\mathrm{d}t + [\cdots]\mathrm{d}\tilde{W} \\
&= \left[sg_t + rs[g - yg_y] + rxg_y + \tfrac{1}{2}\sigma^2 y^2 sg_{yy} + \tfrac{1}{2}\sigma^2\gamma^2 sg_{yy} - \sigma^2 ys\gamma g_{yy}\right]\mathrm{d}t + [\cdots]\mathrm{d}\tilde{W} \\
&= \left[sg_t + rs[g - yg_y] + r\underbrace{x}_{=sy}g_y + \tfrac{1}{2}\sigma^2 sg_{yy}[y^2 + \gamma^2 - 2y\gamma]\right]\mathrm{d}t + [\cdots]\mathrm{d}\tilde{W} \\
&= \left[s\left[\underbrace{g_t + \tfrac{1}{2}\sigma^2 g_{yy}(y-\gamma)^2}_{=0 \text{ by Vecer Thm}}\right] + r\underbrace{sg}_{=v}\right]\mathrm{d}t + [\cdots]\mathrm{d}\tilde{W} \\
&= rv\mathrm{d}t + [\cdots]\mathrm{d}\tilde{W}.
\end{aligned}$$

In conclusion,

$$\mathrm{d}e^{-rt}v = [\cdots]\mathrm{d}\tilde{W}(t).$$

Hence, $e^{-rt}v$ is a martingale under $\tilde{\mathbb{P}}$.

- Note that $e^{-rt}v(t,s,x)$ is a martingale under $\tilde{\mathbb{P}}$.

  Continuing,

$$\mathrm{d}v = s\mathrm{d}g + g\mathrm{d}s + \mathrm{d}s\mathrm{d}g.$$

Recall that

$$\mathrm{d}s = rs\mathrm{d}t + \sigma s\mathrm{d}\tilde{W}, \quad \mathrm{d}g = \sigma(\gamma - y)g_y\mathrm{d}\tilde{W}^S.$$

Here $\mathrm{d}\tilde{W}^S - \mathrm{d}\tilde{W} = \sigma\mathrm{d}t$. Therefore,

$$\begin{aligned}
\mathrm{d}v &= s\mathrm{d}g + g\mathrm{d}s + \mathrm{d}s\mathrm{d}g \\
&= s\sigma(\gamma - y)g_y\mathrm{d}\tilde{W} - \sigma^2 s(\gamma - y)g_y\mathrm{d}t \\
&+ grs\mathrm{d}t + g\sigma s\mathrm{d}\tilde{W} \\
&+ \sigma^2 s(\gamma - y)g_y\mathrm{d}t \\
&= rv\mathrm{d}t + s\sigma\left[(\gamma - y)g_y + g\right]\mathrm{d}\tilde{W}
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{d}e^{-rt}v &= e^{-rt}\left[\mathrm{d}v - rv\mathrm{d}t\right] \\
&= e^{-rt}\left[rv\mathrm{d}t + s\sigma\left[(\gamma - y)g_y + g\right]\mathrm{d}\tilde{W}\right]
\end{aligned}$$

Denote the hedging portfolio's value by $M(t)$. So

$$\mathrm{d}M(t) = \Delta(t)\mathrm{d}S(t) + r(M(t) - \Delta(t)S(t))\mathrm{d}t = rM(t)\mathrm{d}t + \sigma\Delta(t)S(t)\mathrm{d}\tilde{W}(t)$$

Thus,

$$\mathrm{d}e^{-rt}M(t) = e^{-rt}\left(\mathrm{d}M(t) - rM(t)\mathrm{d}t\right) = e^{-rt}\sigma\Delta(t)S(t)\mathrm{d}\tilde{W}(t)$$

We need to have

$$\mathrm{d}e^{-rt}M = \mathrm{d}e^{-rt}v$$

Which holds iff

$$\sigma\Delta s\mathrm{d}\tilde{W} = s\sigma\left[(\gamma - y)g_y + g\right]\mathrm{d}\tilde{W}$$

Thus, it suffices to let

$$\begin{aligned}
\Delta(t) &= (\gamma(t) - Y(t))g_y(t, Y(t)) + g(t, Y(t)) \\
&= \gamma(t)g_y(t, Y(t)) + \underbrace{g(t, Y(t)) - Y(t)g_y(t, Y(t))}_{=v_s(t, S(t), Y(t))} \\
&= \gamma(t)v_x(t, S(t), X(t)) + v_s(t, S(t), X(t))
\end{aligned}$$