Exercise 8.1 (Determination of L_* by smooth pasting)

Recall that

$$v_L(x) = \begin{cases} K - x & 0 \le x \le L \\ (K-L)L^{\frac{2r}{\sigma^2}} \cdot x^{-\frac{2r}{\sigma^2}} & x \ge L. \end{cases}$$

Show that smooth pasting (*i.e.*, $v_L'(L-) = v_L'(L+)$) is satisfied only at L_* where

$$L_* := \frac{2r}{2r+\sigma^2} \cdot K$$

Proof

Notice that $v_L'(L-) = -1$ and

$$\begin{aligned} v_L'(L+) &= -(K-L)L^{\frac{2r}{\sigma^2}} \cdot \frac{2r}{\sigma^2} \cdot L^{-\frac{2r+\sigma^2}{\sigma^2}} \\ &= -\frac{K-L}{L} \cdot \frac{2r}{\sigma^2}. \end{aligned}$$

Thus,

$$\begin{aligned} v_L'(L+) = v_L'(L-) &\iff \frac{K-L}{L} = \frac{\sigma^2}{2r} \\ &\iff \frac{K}{L} = \frac{\sigma^2 + 2r}{2r} \\ &\iff L = \frac{2r}{2r+\sigma^2} \cdot K \end{aligned}$$