Exercise 8.2

Consider two different strike prices $K_1 < K_2$ and let $v_1(x), v_2(x)$ be their respective prices. Show that $v_2(x)$ satisfies the first two bounds from linear complimentarity conditions, but not the last one. Namely, $v_2(x)$ satisfies

- $\forall x \geq 0, v_2(x) \geq (K_1 - x)^+$
- $\forall x \geq 0, rv_2(x) - rxv_2'(x) - \frac{1}{2}\sigma^2 x^2 v_2''(x) \geq 0$
- There exists $x \geq 0$ such that neither of these two bounds holds with equality

Proof

Note that

$$(K_2 - x)^+ - (K_1 - x)^+ = \begin{cases} K_2 - K_1 & \text{if } x < K_1 \\ (K_2 - x)^+ & \text{if } x \geq K_1. \end{cases}$$

Therefore $(K_2 - x)^+ \geq (K_1 - x)^+$. Now since $v_2(x) \geq (K_2 - x)^+$, the first bounds holds. The second bound always holds for $v_{L_*}(x)$ for an arbitrary K. It remains to show the third condition is violated for v_2. Recall that

$$L_*(K) = \frac{2r}{2r + \sigma^2} K.$$

Therefore, there exists x s.t. $L_*(K_1) < x < L_*(K_2)$. We have that

$$v_2(x) = (K_2 - x)^+ = K_2 - x.$$

Here we used that $L_*(K) < K$ for an arbitrary K. Therefore,

$$rv_2(x) - rxv_2'(x) - \tfrac{1}{2}\sigma^2 x^2 v_2''(x) = r(K_2 - x) - rx(-1) - 0 = rK_2.$$

Thus

$$\begin{aligned} v_2(x) &> 0 \\ rv_2(x) - rxv_2'(x) - \tfrac{1}{2}\sigma^2 x^2 v_2''(x) &> 0 \end{aligned}$$