Exercise 8.3 (Solving the linear complimentary conditions)

Suppose that the bounded continuous function $v(x)$ satisfies the linear complimentary conditions. Namely,

- $v(x) \geq (K-x)^+$ for all $x \geq 0$
- $rv(x) - rxv'(x) - \frac{1}{2}\sigma^2 x^2 v''(x) \geq 0$ for all $x \geq 0$
- At least one of the above two inequalities holds with equality for each $x \geq 0$

Show that $v = v_{L^*}$ where

$$L^* = \frac{2r}{2r+\sigma^2}K$$

Proof

We begin by showing that if x satisfies second-order bound with equality over some closed interval I, then for some constant A and B it must hold that

$$v(x) = Ax^{-\frac{2r}{\sigma^2}} + Bx \quad \forall x \in I.$$

Indeed, the following linear space

$$\mathcal{L}_I := \{rv_2(x) - rxv_2'(x) - \tfrac{1}{2}\sigma^2 x^2 v_2''(x) = 0 \quad \forall x \in I\}$$

has a basis of the form $\{x^{p_1}, x^{p_2}\}$. It suffices to show that $\{p_1, p_2\} = \{-\frac{2r}{\sigma^2}, 1\}$ since for $p_1 \neq p_2$, x^{p_1} and x^{p_2} are clearly linearly independent. If $x^p \in \mathcal{L}_I$, then

$$rx^p - rpx^p - \frac{1}{2}\sigma^2 p(p-1)x^p = 0 \quad \forall x \in I.$$

Therefore, $r - rp - \frac{1}{2}\sigma^2 p(p-1) = 0$. $p = 1$ and $p = \frac{-2r}{\sigma^2}$ are roots of this second order equation. Now suppose that $I = [x_1, x_2]$ with $0 < x_1 < x_2 < +\infty$ and also

$$v(x) = (K-x_1)^+,\ \forall x \in (x_1 - \epsilon, x_1] \text{ and } v(x) = (K-x_2)^+,\ \forall x \in [x_2, x_2+\epsilon)$$

Since v is continuous, it holds that

$$Ax_i^{-\frac{2r}{\sigma^2}} + Bx_i = (K-x_i)^+.$$

Moreover,

$$\begin{aligned} v'(x_i) = \frac{-2rAx_i^{-\frac{2r}{\sigma^2}}}{\sigma^2 x_i} + B &= \frac{-2r\left[(K-x_i)^+ - Bx_i\right]}{\sigma^2 x_i} + B \\ &= \frac{-2r\left[(\frac{K}{x_i}-1)^+ - B\right]}{\sigma^2} + B \\ &= \frac{\left(\frac{K}{x_i}-1\right)^+ - B - \frac{\sigma^2 B}{2r}}{-\frac{\sigma^2}{2r}} \end{aligned}$$

Case A: $K < x_2$ In this case,

$$v'(x_2) = \frac{\left(\frac{K}{x_2} - 1\right)^+ - B - \frac{\sigma^2 B}{2r}}{-\frac{\sigma^2}{2r}}$$
$$= \frac{-B - \frac{\sigma^2 B}{2r}}{-\frac{\sigma^2}{2r}}$$
$$= 0.$$

Thus, $B = 0$. Consequently, since $Ax_2^{-\frac{2r}{\sigma^2}} + Bx_2 = (K - x_2)^+$, we conclude that $Ax_2^{-\frac{2r}{\sigma^2}} = 0$. This in turn yields that $A = 0$ which means $v \equiv 0$ on I.

Case B: $x_2 \le K$ In this case,

$$v'(x_i) = \frac{\frac{K}{x_i} - 1 - B - \frac{\sigma^2 B}{2r}}{-\frac{\sigma^2}{2r}}$$
$$= -1.$$

This means that $x_1 = x_2$ which is a contradiction. Next, since v is bounded, the second-order bound cannot be satisfied with equality for any interval of the form $[0, x_2]$. To see this, note that if $A \neq 0$, then

$$\lim_{x \downarrow 0} v(x) = +\infty \text{ or } -\infty$$

On the other hand, if $A = 0$, then $v(0) = 0 \geq K$. This is contradiction. The same argument holds if $x_2 = +\infty$. Next, note that $v(x) = (K - x)^+$ cannot hold for all $x \geq 0$ since v' is continuous.

Since v is continuous and the set of x where the first holds with *strict* inequality (and hence the second bound holds with equality) must be union of disjoint open intervals, a quick check shows that the only option which has not been ruled out yet must be

First bound holds with equality on $[0, x_1]$ only for some $x_1 > 0$!

Thus

$$\forall x \in [0, x_1], \qquad v(x) = (K - x)^+$$
$$\forall x \in (x_1, +\infty), \qquad v(x) = Ax^{-\frac{2r}{\sigma^2}} + Bx$$

Since v' is continuous on $[0, x_1]$, we must have that $x_1 \leq K$ as otherwise v' will be discontinuous at $x = K$. Hence,

$$\forall x \in [0, x_1], \qquad v(x) = K - x$$
$$\forall x \in (x_1, +\infty), \qquad v(x) = Ax^{-\frac{2r}{\sigma^2}} + Bx$$

Note that

$$B \neq 0 \Rightarrow \lim_{x \to +\infty} |v(x)| = +\infty.$$

But since v is bounded, B must be equal to zero. We have that

$$\begin{aligned} v'(x_1-) &= -1 \\ &= -\frac{2r}{\sigma^2} \cdot \frac{Ax_1^{-\frac{2r}{\sigma^2}}}{x_1} \\ &= v'(x_1+) \end{aligned}$$

Moreover,

$$\begin{aligned} v(x_1) &= K - x_1 \\ &= Ax_1^{-\frac{2r}{\sigma^2}} \\ &= \lim_{x \downarrow x_1} v(x) \end{aligned}$$

Thus,

$$\begin{aligned} Ax_1^{-\frac{2r}{\sigma^2}} &= \frac{\sigma^2}{2r} \cdot x_1 \\ &= K - x_1. \end{aligned}$$

This immediately yields that $x_1 = L_*$ and $A = (K - L_*)L_*^{\frac{2r}{\sigma^2}}$.